電子回路論第10回
Electric Circuits for Physicists

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Comment: Impedance match/mismatch

Propagation of a wave:

Impedance match: complete absorption (propagation without reflection)

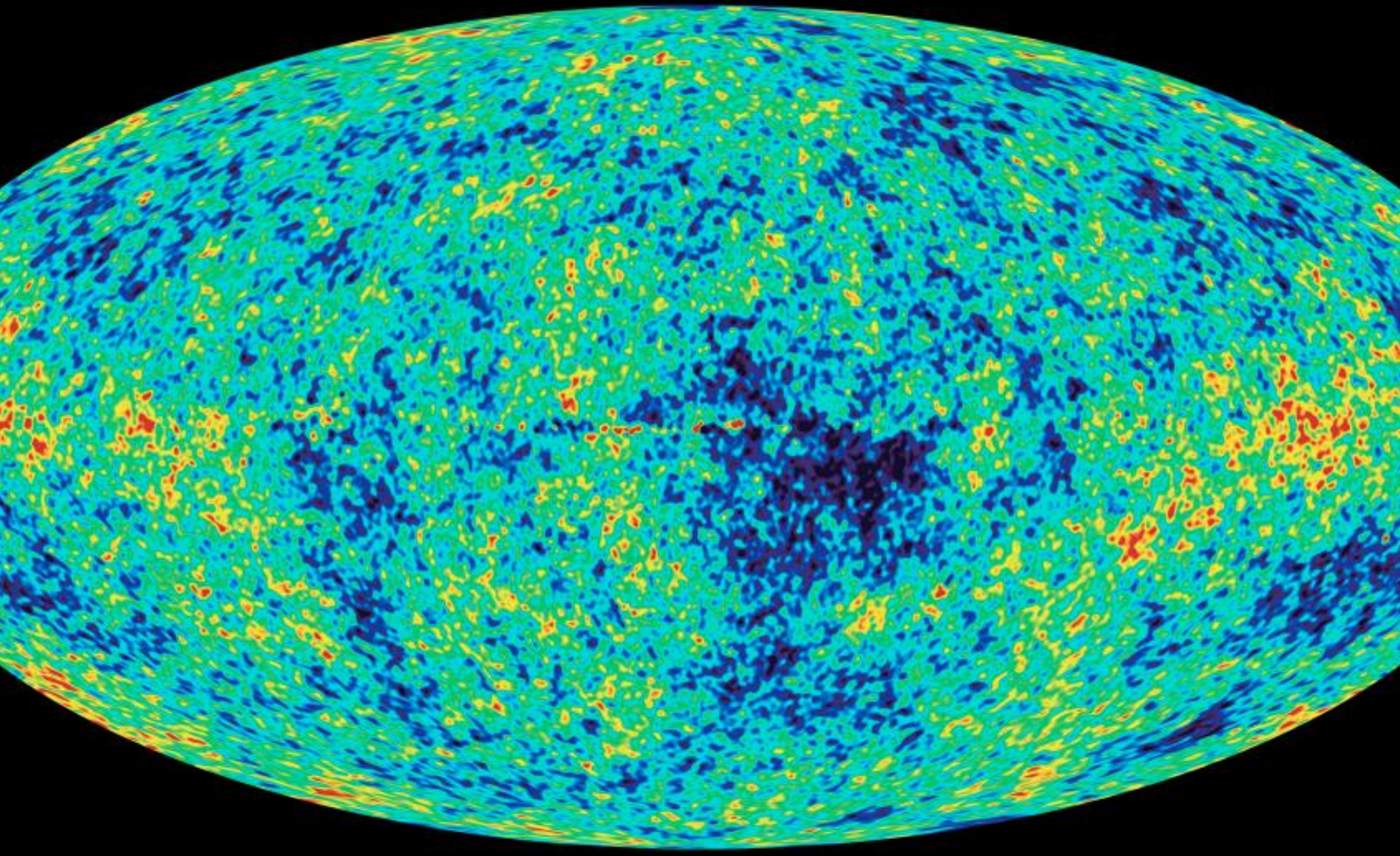
Mismatch: wave reflection

Impedance match/mismatch is an important concept applicable to a broad area of physics.

- Antenna: should be matched to the vacuum.

  EM wave propagation simulation: boundary is shunted with the characteristic impedance of vacuum.
- Optics: impedance mismatch → disagreement in refractive index
- Plasma: should be matched to electrodes for excitation.
- Phonon impedance mismatch at low temperatures: Kapitza resistance
- Sound insulated booth: should have sound impedance mismatch.

Ch6. Noises and Signals



Outline

6.1 Fluctuation
- 6.1.1 Fluctuation-Dissipation theorem
- 6.1.2 Wiener-Khintchine theorem
- 6.1.3 Noises in the view of circuits
- 6.1.4 Nyquist theorem
- 6.1.5 Shot noise
- 6.1.6 1/f noise
- 6.1.7 Noise units
- 6.1.8 Other noises

6.2 Noises from amplifiers
- 6.2.1 Noise figure
- 6.2.2 Noise impedance matching

Noises

Electric circuits transform: 1) Information

2) Electromagnetic power

on some physical quantities like voltages, current, …

Noises: stochastic (uncontrollable, unpredictable by human) variation in other words, fluctuation in such a quantity.

Internal noise

Intrinsic noise: Thermal noise (Johnson-Nyquist noise), Shot noise

Noise related to a specific physical phenomenon

Avalanche, Popcorn, Barkhausen, etc.

1/f noise: Name for a group of noises with spectra 1/f.

External noise

EMI, microphone noise, etc.

6.1 Fluctuation

Quantity x, fluctuation $\delta x = x - \bar{x}$

$$\overline{(\delta x)^2} = \overline{(x-\bar{x})^2} = \overline{x^2} - \bar{x}^2 \quad (\overline{\delta x} = 0)$$

$g(x)$: distribution function of x

Fourier transform: $$u(q) = \mathscr{F}\{g(x)\} = \int_{-\infty}^{\infty} g(x) e^{ixq} \frac{dx}{\sqrt{2\pi}}$$

$u(q)$: **characteristic function** of the distribution

From Taylor expansion, any moment can be obtained as

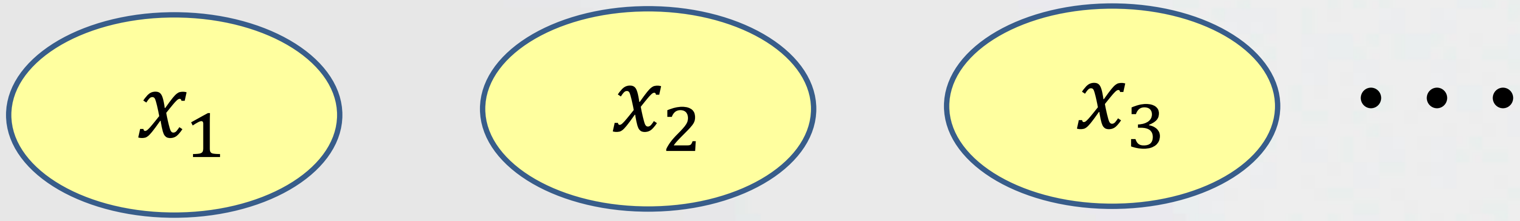
$$\overline{x^n} = \frac{\sqrt{2\pi}}{i^n}\left[\frac{d^n}{dq^n}u(q)\right]_{q=0}$$

Moments to high orders → reconstruction of $g(x)$

6.1 Fluctuation

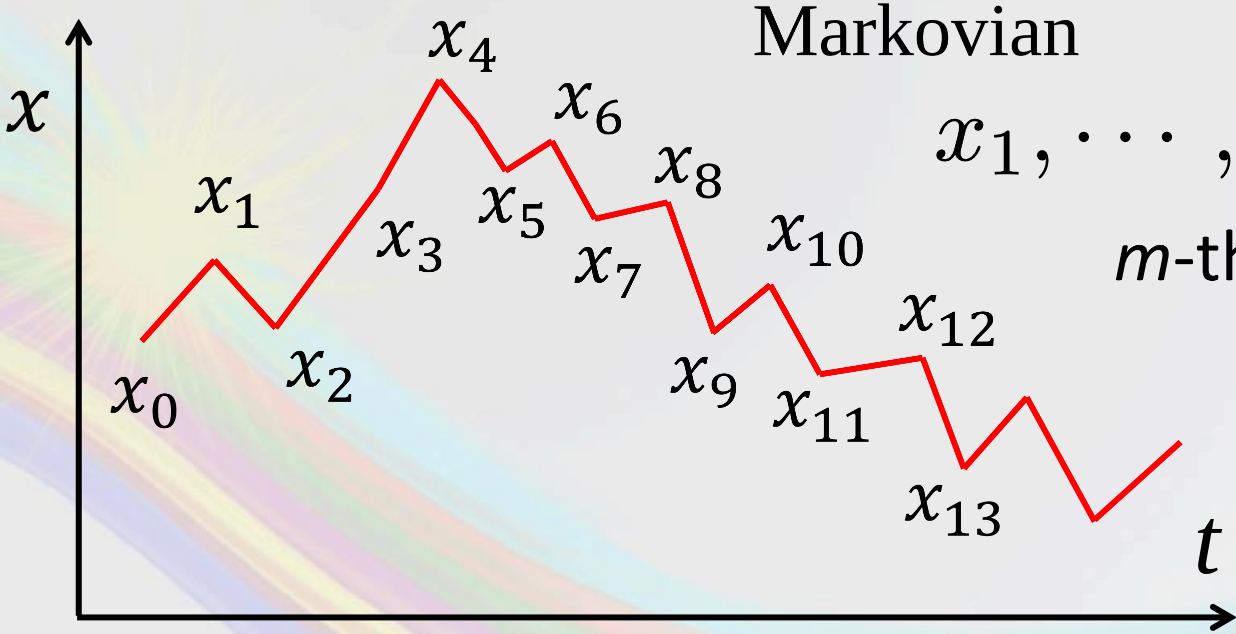
In electric circuits we need to consider two kinds of averages:

substance 1 substance 2 substance 3



x_j: independent

Ensemble average: $\bar{x}$



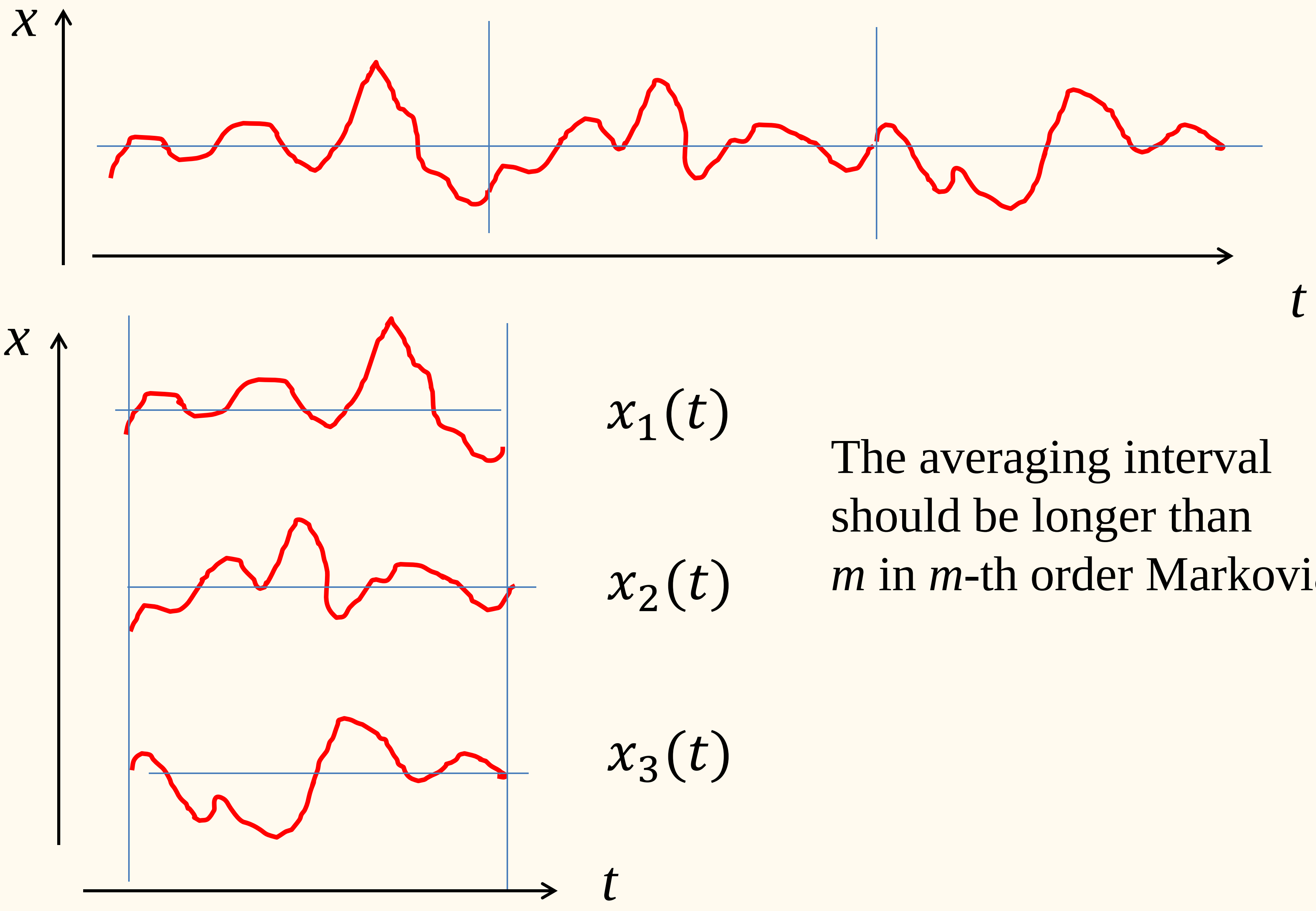
Markovian

$$x_1, \cdots, x_m \xrightarrow{\text{affect}} x_{m+1}$$

m-th order Markovian

Time average of fluctuating variable: $\langle x \rangle$

Random process to distribution



The averaging interval should be longer than *m* in *m*-th order Markovian.

6.1.1 Fluctuation-Dissipation Theorem

久保亮五
Ryogo Kubo 1920-1995

Harry Nyquist
1889-1976

Nobert Wiener
1894-1964

Aleksandr Khinchin
1894-1959

Power Spectrum

Consider probability sets in the interval [0,T).

set index: j $$x_j(t) = \sum_{n=1}^{\infty} (a_{jn} \cos \omega_n t + b_{jn} \sin \omega_n t), \quad \omega_n = \frac{2n\pi}{T}$$

$$\mathscr{P}_{jn} = (a_{nj} \cos \omega_n t + b_{nj} \sin \omega_n t)^2$$ (Power)

$$\langle \mathscr{P}_n \rangle = \frac{1}{2} \langle a_n^2 + b_n^2 \rangle$$ $\because$ cross product terms are averaged out

---

Random process:
Gaussian distribution in time

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[-\frac{x^2}{2\sigma^2} \right]$$

$$\overline{(\delta x)^2} = \sigma^2, \quad \overline{\left(\sum_{j=1}^{m} \delta x_j \right)^2} = m\sigma^2$$

Then $\overline{\langle \mathscr{P}_n \rangle} = \sigma_n^2$ (non-Markovian)

Power Spectrum

Power spectrum $G(\omega)$

Frequency band width $\delta\omega$: separation between two adjacent frequencies

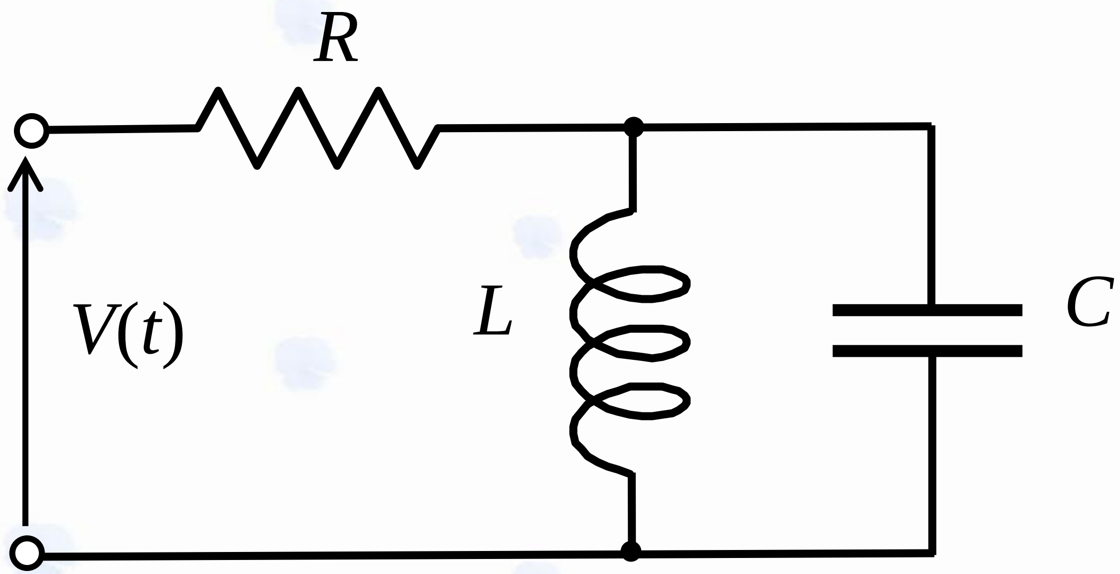
$$\delta\omega = \omega_{n+1} - \omega_n = \frac{2(n+1)\pi}{T} - \frac{2n\pi}{T} = \frac{2\pi}{T}$$

$$G(\omega_n)\frac{\delta\omega}{2\pi} = \overline{\langle \mathscr{P}_n \rangle}\ (= \sigma_n^2)$$

$$\overline{\langle x^2(t) \rangle} = \sum_{n=1}^{\infty} \overline{\langle \mathscr{P}_n \rangle} \quad (\overline{x(t)} = 0)$$

$$= \sum_n G(\omega_n)\frac{\delta\omega}{2\pi} \to \int_0^{\infty} G(\omega)\frac{d\omega}{2\pi}$$

6.1.1 Fluctuation-Dissipation Theorem



$$\omega_0 \equiv 1/\sqrt{LC}$$

$$Z(i\omega) = \frac{R(\omega_0^2 - \omega^2) + i\omega_0^2\omega L}{\omega_0^2 - \omega^2},$$

$$Y(i\omega) = \frac{\omega_0^2 - \omega^2}{R(\omega_0^2 - \omega^2) + i\omega_0^2\omega L}$$

Johnson-Nyquist noise

$V(t)$ noise power spectrum $\rightarrow G_v(\omega)$

$$G_v(\omega) = 4k_{\mathrm{B}}T\mathrm{Re}[Z(i\omega)]$$

$$G_v(\omega) = 4k_{\mathrm{B}}TR$$

Johnson-Nyquist noise
Thermal noise

White noise

One representation of the fluctuation-dissipation theorem

6.1.2 Wiener-Khintchine Theorem

Autocorrelation function $C(\tau) = \overline{\langle x(t)x(t+\tau)\rangle}$

$$= \overline{\sum_{n,m} \langle [a_n \cos\omega_n t + b_n \sin\omega_n t][a_m \cos\omega_m (t+\tau) + b_m \sin\omega_m (t+\tau)]\rangle}$$

$$= \frac{1}{2}\sum_n \overline{\langle a_n^2 + b_n^2\rangle} \cos\omega_n \tau = \sum_n \overline{\langle \mathscr{P}_n\rangle} \cos\omega_n \tau$$

$$= \int_0^\infty G(\omega) \cos\omega\tau \frac{d\omega}{2\pi}$$

$$C(\tau) = \int_0^\infty G(\omega) \cos\omega\tau \frac{d\omega}{2\pi}$$

$$G(\omega) = 4\int_0^\infty C(\tau) \cos\omega\tau \, d\tau$$

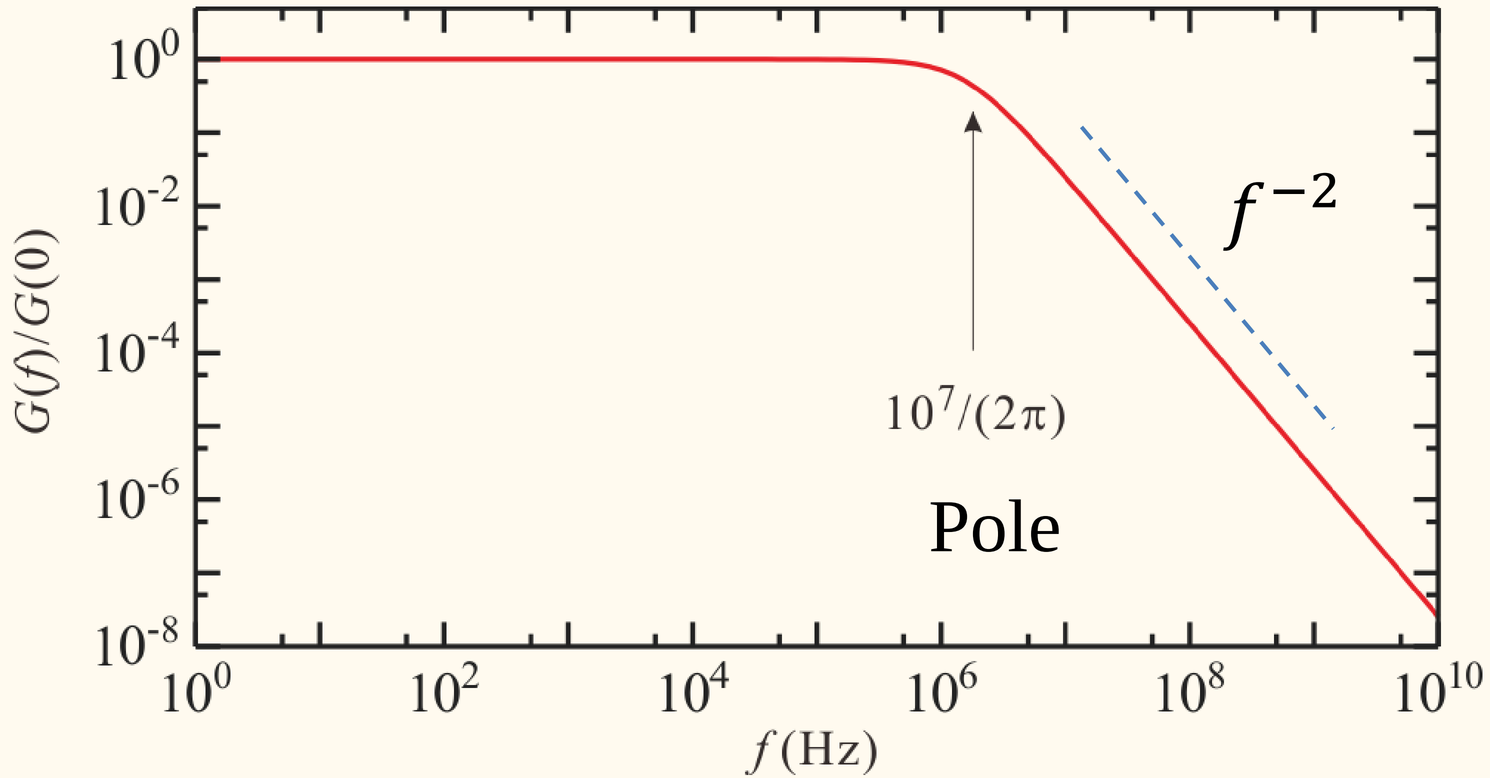
Wiener-Khintchine theorem

6.1.2 Wiener-Khintchine Theorem

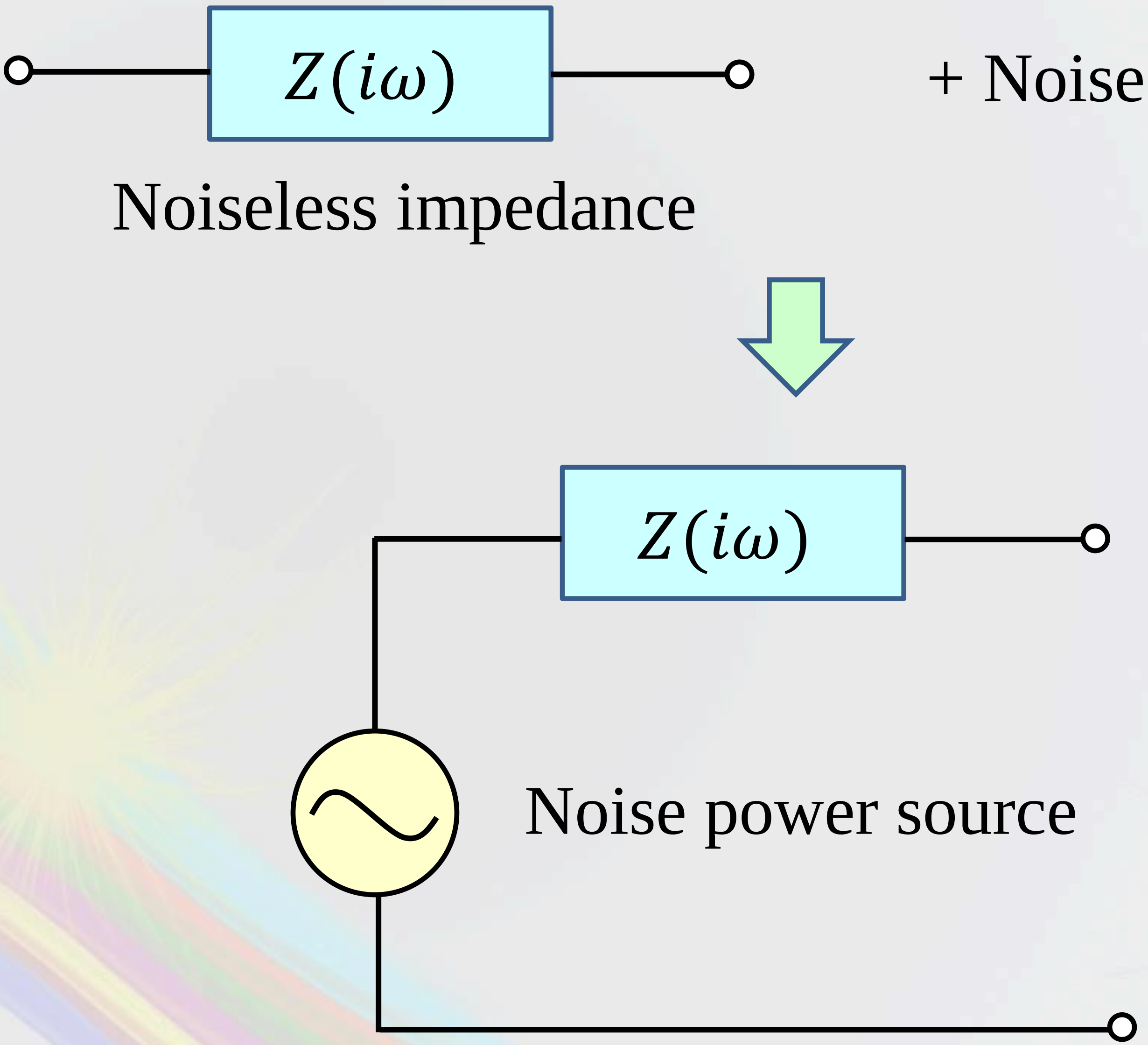
Example) $C(\tau) = \exp\left(-\frac{\tau}{\tau_0}\right)$

$$G(f) = 4\int_0^\infty e^{-\tau/\tau_0}\cos(2\pi f\tau)d\tau = \frac{4\tau_0}{1+(2\pi f\tau_0)^2}$$

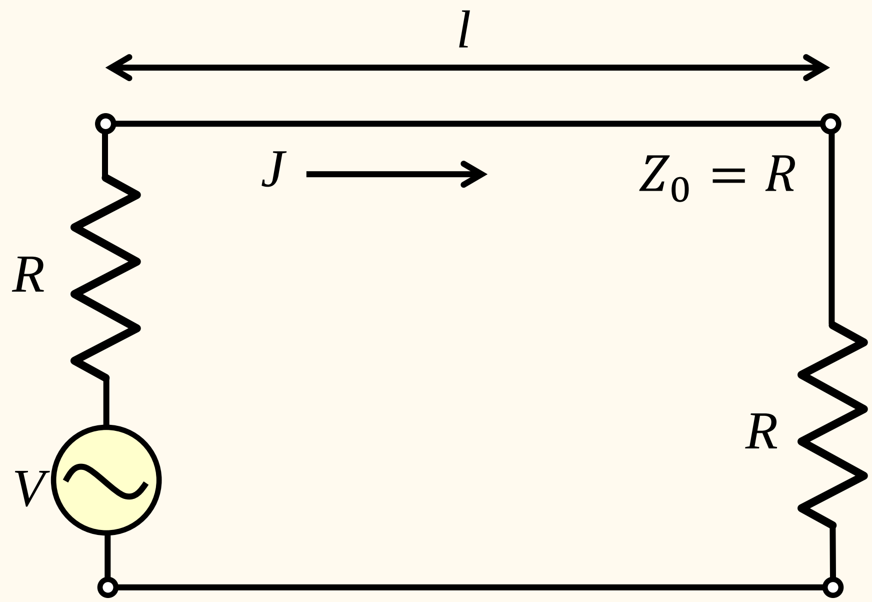
$\tau_0 = 10^{-7}\mathrm{s}$ (10MHz)



6.1.3 Electric circuit treatment of noise



6.1.4 Nyquist Theorem



Mode density on a transmission line with length *l*

$$\omega_n = c^* k_n = \frac{2\pi n c^*}{l} \qquad \therefore \quad \delta\omega = \frac{2\pi c^*}{l}$$

Bidirectional → Freedom × 2

Bose distribution $$f(\hbar\omega, T) = \frac{1}{\exp(\hbar\omega / k_{\mathrm{B}} T) - 1}$$

6.1.4 Nyquist Theorem

Thermal energy density per freedom

$$\frac{\hbar\omega}{\exp(\hbar\omega/k_\mathrm{B}T)-1} \sim \frac{\hbar\omega}{1+(\hbar\omega/k_\mathrm{B}T)-1} = k_\mathrm{B}T \quad (k_\mathrm{B}T \gg \hbar\omega)$$

Thermal energy density in band $\Delta\omega$

$$2\frac{\Delta\omega}{\delta\omega}k_\mathrm{B}T = \frac{2k_\mathrm{B}Tl}{2\pi c^*}\Delta\omega$$, a half of which flows in one-direction

Energy flowing out from the end:

$$\frac{k_\mathrm{B}Tl}{2\pi c^*}\Delta\omega \times \frac{1}{l} \times c^* = k_\mathrm{B}T\Delta f \quad (2\pi f = \omega)$$

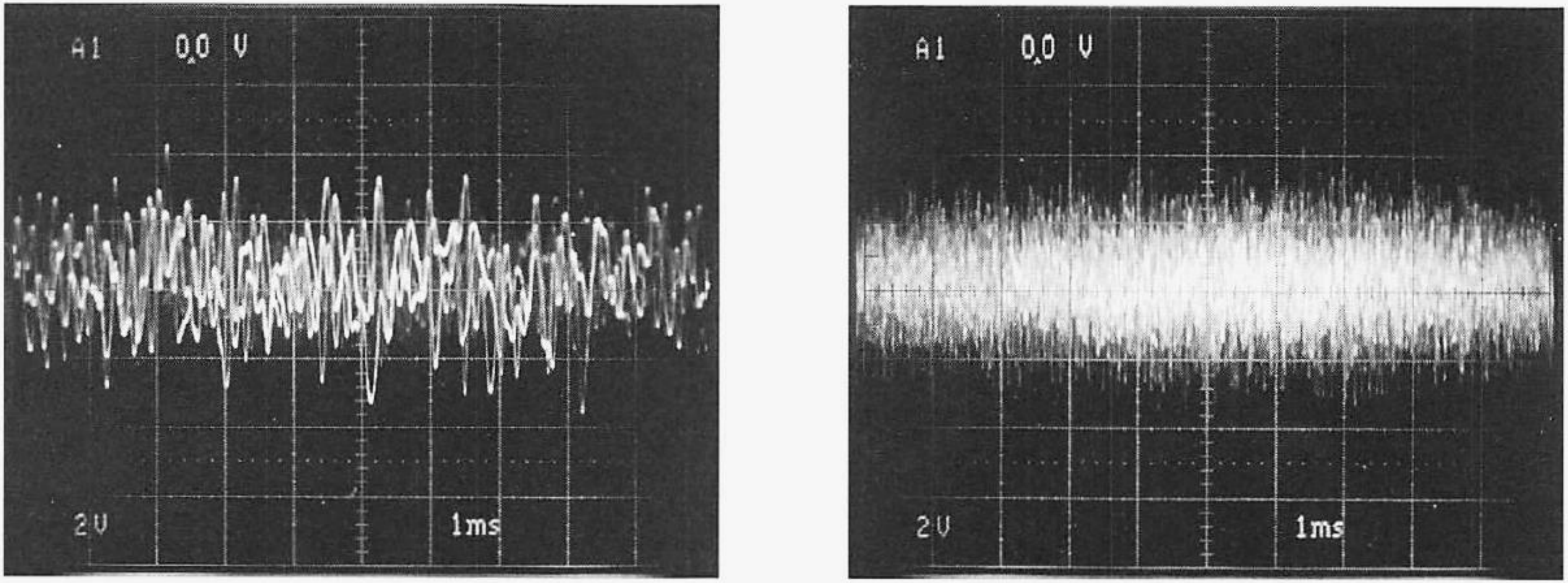
equals the energy supplied from the noise source.

$$\overline{J^2}R = k_\mathrm{B}T\Delta f, \quad \overline{V^2} = 4Rk_\mathrm{B}T\Delta f \quad (V = 2RJ)$$

$$\sqrt{\overline{J^2V^2}} = 2k_\mathrm{B}T\Delta f$$

→ Noise Temperature

Thermal noise



(a) 上限周波数5 kHz (−3dB) 1 V_{rms}の熱雑音を1 ms/divで観測

(b) 上限周波数100 kHz (−3dB)1 V_{rms}の熱雑音を1 ms/divで観測

〈写真 1-1〉 熱雑音の測定

Single Electron

Time domain: δ-function approximation

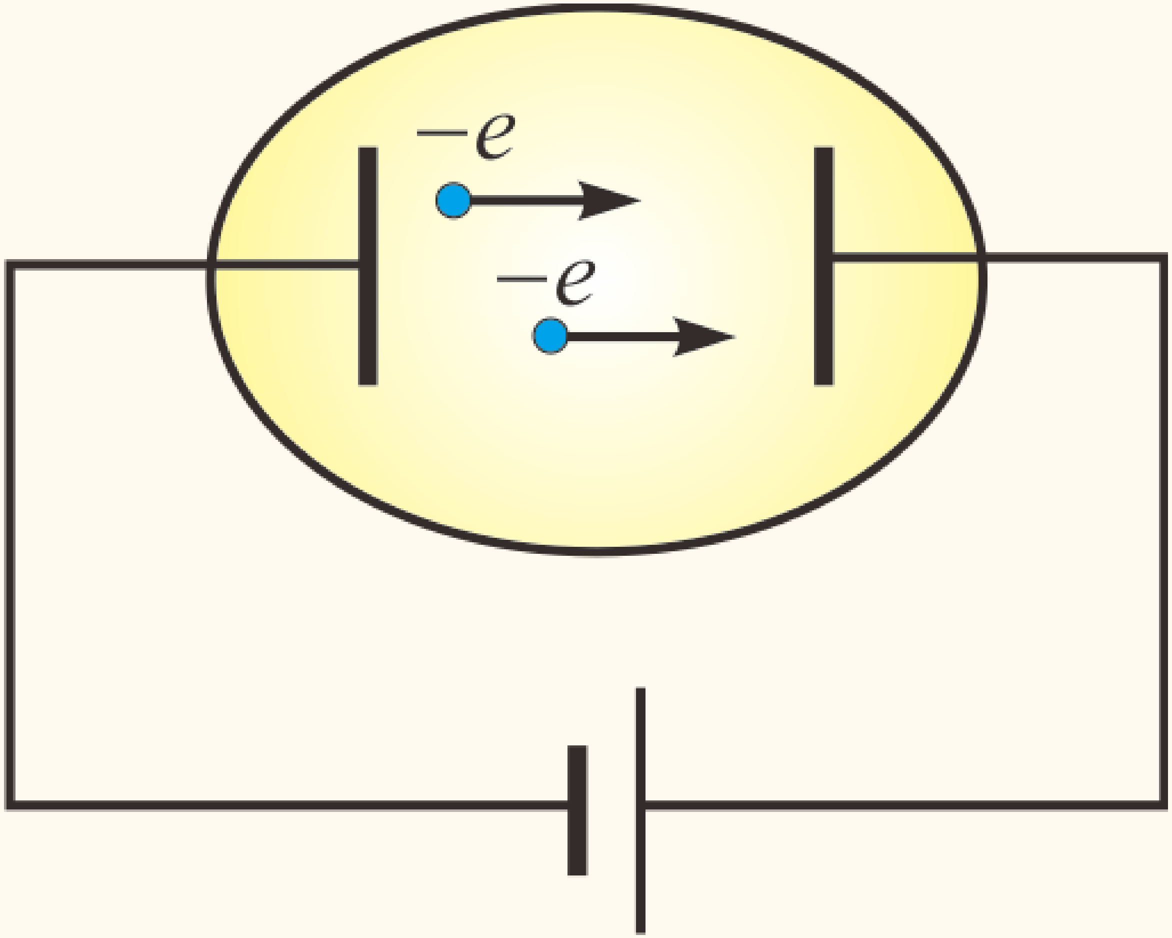
$$J_e(t) = e\delta(t - t_0)$$
$$= e\int_{-\infty}^{\infty} e^{2\pi i f(t-t_0)} df = 2e\int_0^{\infty} \cos\left[2\pi f(t - t_0)\right] df$$

Uniform $2e$ in frequency domain: fluctuation at each frequency
Coherent only at $t = t_0$

Current fluctuation density for infinitesimal band df

$$\delta J = d\sqrt{\langle J_e^2 \rangle} = \frac{2e}{\sqrt{2}} df = \sqrt{2} e df$$

6.1.5 Shot Noise



6.1.5 Shot Noise

Double Electron

$$\overline{\langle \delta j^2 \rangle} = (j_p + j_q e^{i\phi})(j_p + j_q e^{-i\phi}) = j_p^2 + j_q^2 + 2 j_p j_q \cos\phi$$

ϕ: coherent phase shift → averaged out

$$\overline{\langle \delta j^2 \rangle} = j_p^2 + j_q^2 = 2 \times (\sqrt{2}e)^2 df$$

N-Electron

$$\overline{\langle \delta J^2 \rangle} = N \times 2e^2 df = 2e\overline{J}df \quad (\overline{J} = eN)$$

Quantum mechanical correlation → Modification from random

6.1.5 Shot Noise

Example: pn junction

Current-Voltage characteristics: $J(V) = J_0 \left[\exp\left(\frac{eV}{k_\mathrm{B}T}\right) - 1\right]$

Differential resistance $r_\mathrm{d} = \left(\frac{dJ}{dV}\right)^{-1} = \left[\frac{eJ_0}{k_\mathrm{B}T}\exp\left(\frac{eV}{k_\mathrm{B}T}\right)\right]^{-1} = \frac{k_\mathrm{B}T}{e}\frac{1}{J+J_0}$

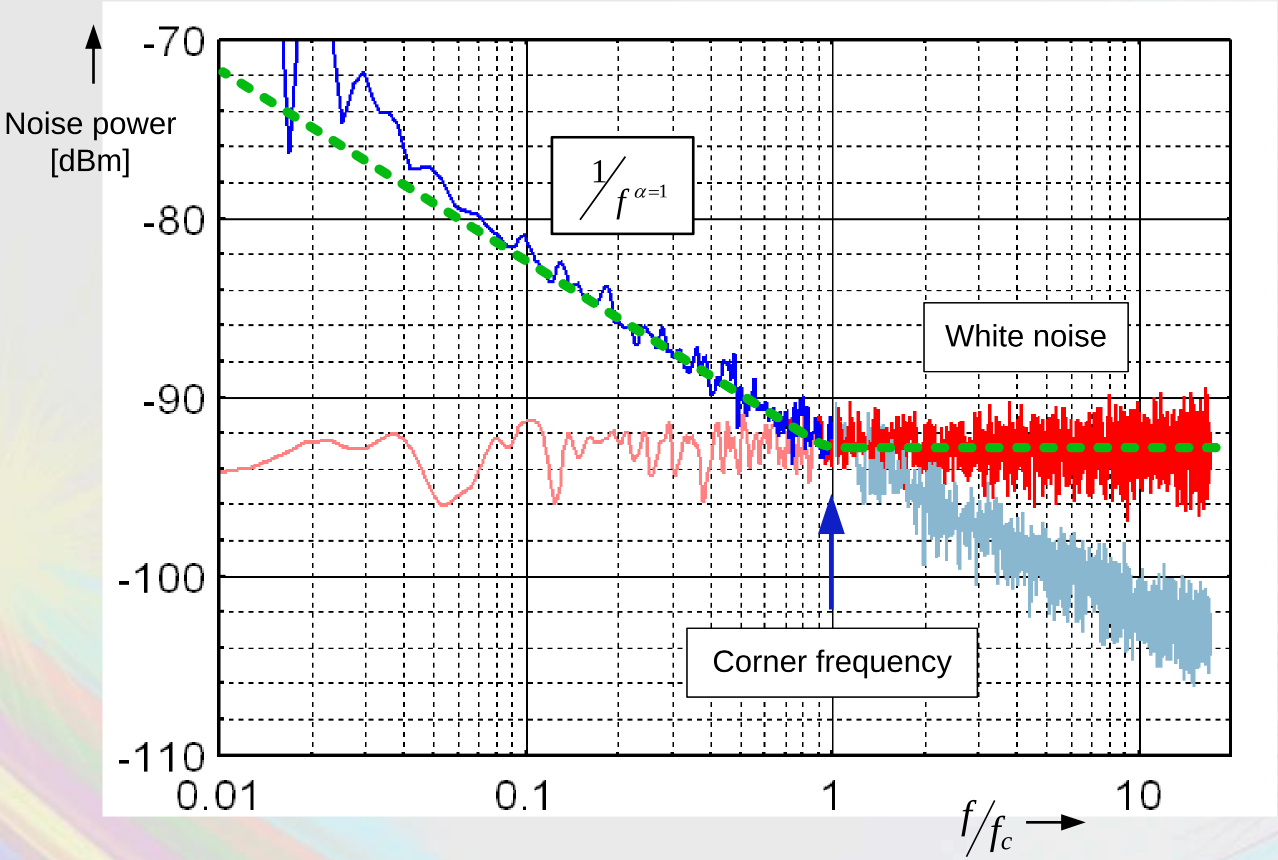
$$J \gg J_0 \;\rightarrow\; r_\mathrm{d} \sim k_\mathrm{B}T/eJ$$

$$\overline{\langle(\delta J)^2\rangle} = 2e\frac{k_\mathrm{B}T}{er_\mathrm{d}}df = 4k_\mathrm{B}T\frac{1}{2r_\mathrm{d}}df$$

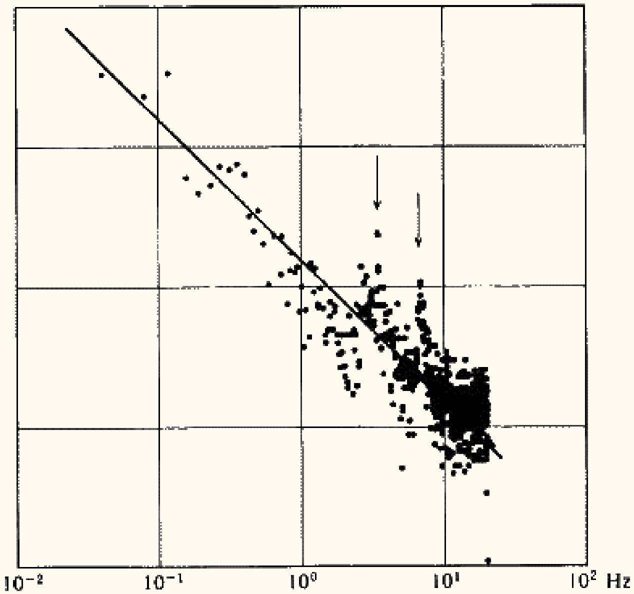
$$(\delta V)^2 = 4\frac{r_\mathrm{d}}{2}k_\mathrm{B}T\Delta f$$

6.1.6 1/f noise

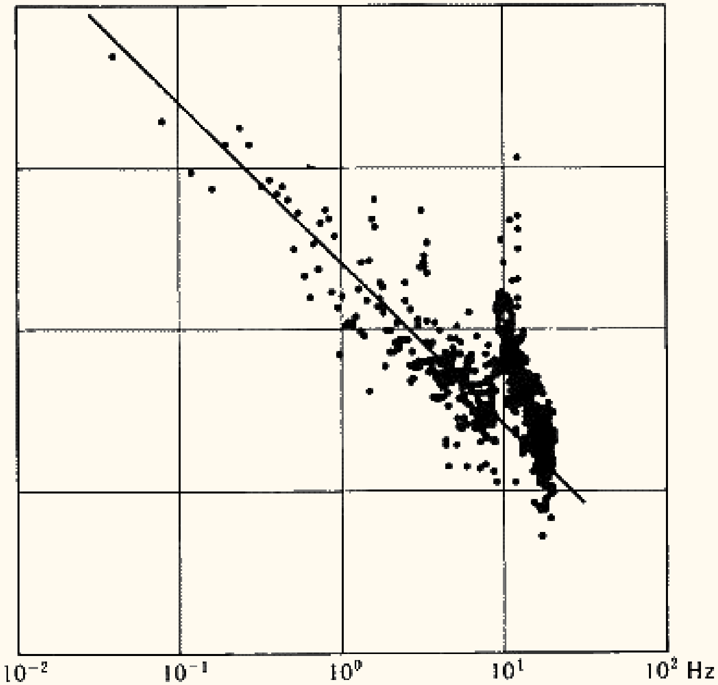
$$(\delta V)^2 = K J^a R^2 \frac{\Delta f}{f^\alpha}$$



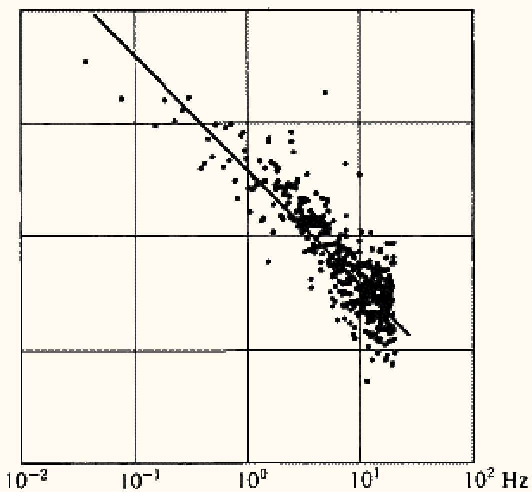
6.1.6 1/f noise



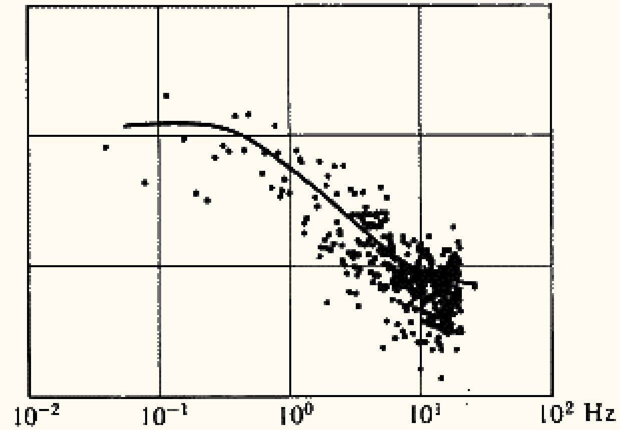
J. S. Bach, Brandenburg Concerto No.1



A. Vivaldi, Four Seasons, Spring



Kawai Naoko, Smile for me



S. Sato, Keshin (incarnation) II

"Unit" of Noise

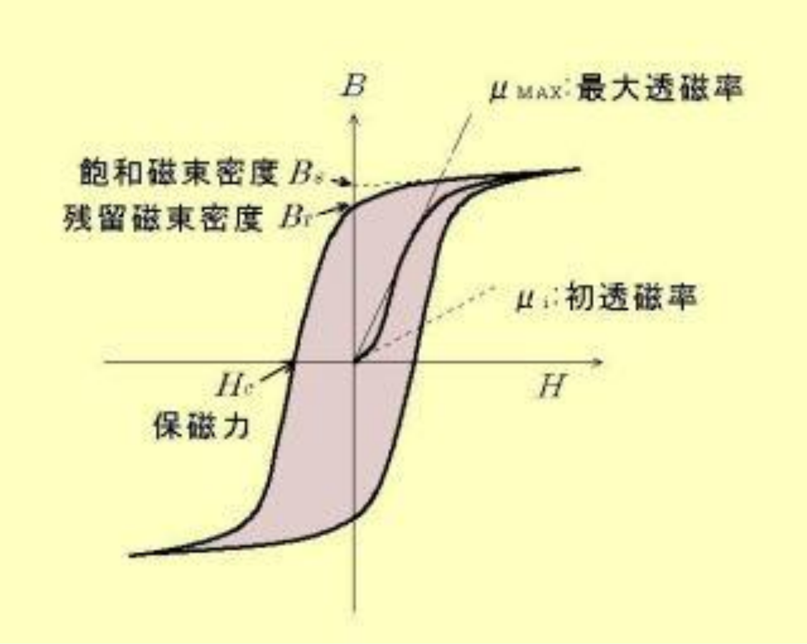
Noise: Power spectrum per frequency

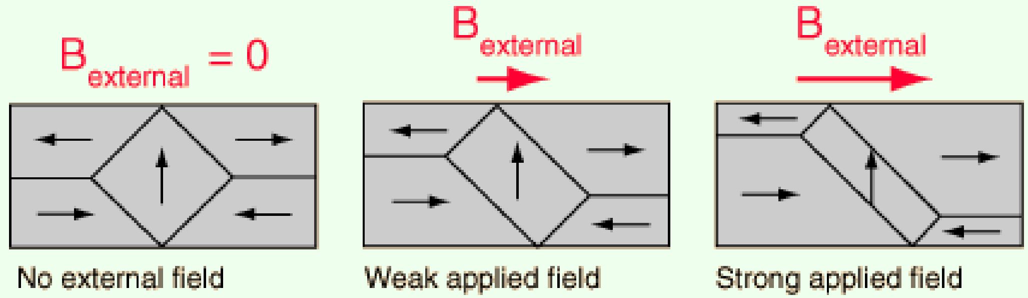
$$\overline{j_n^2} = \overline{\delta J^2}/\Delta f, \quad \overline{e_n^2} = \overline{\delta V^2}/\Delta f$$

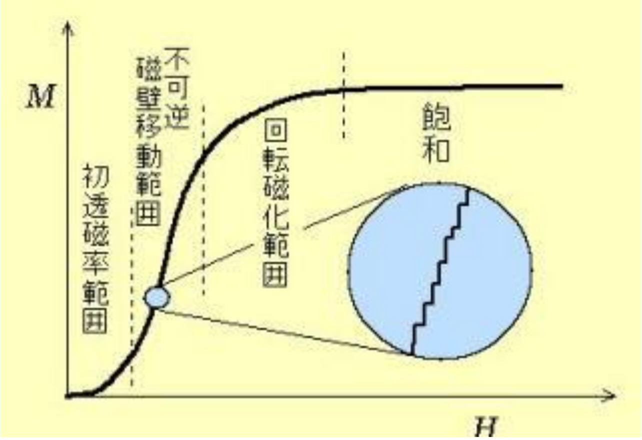
↓ unit of $\sqrt{\overline{j_n^2}}, \quad \sqrt{\overline{e_n^2}}$

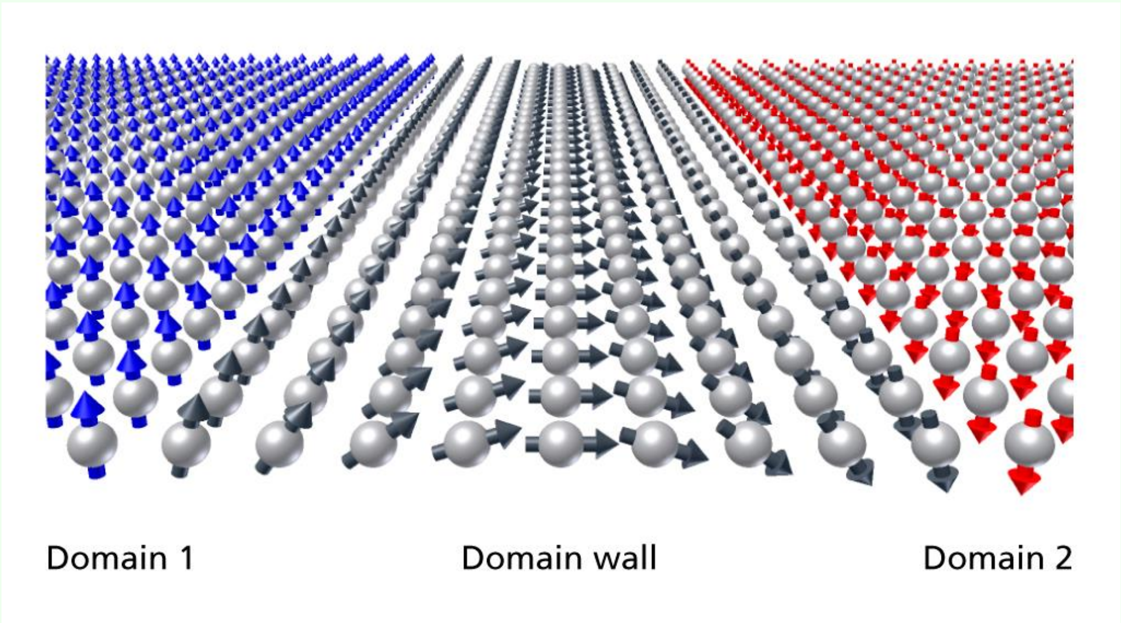
$$\mathrm{A}/\sqrt{\mathrm{Hz}}, \quad \mathrm{V}/\sqrt{\mathrm{Hz}}$$

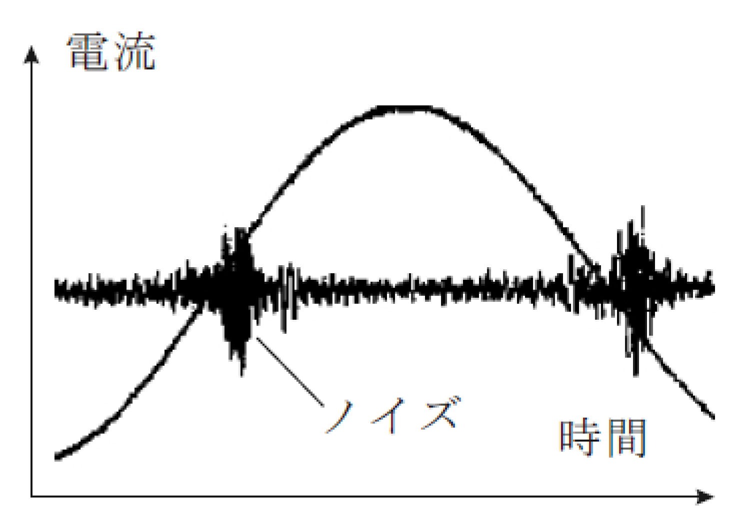
Other noises: Barkhausen noise





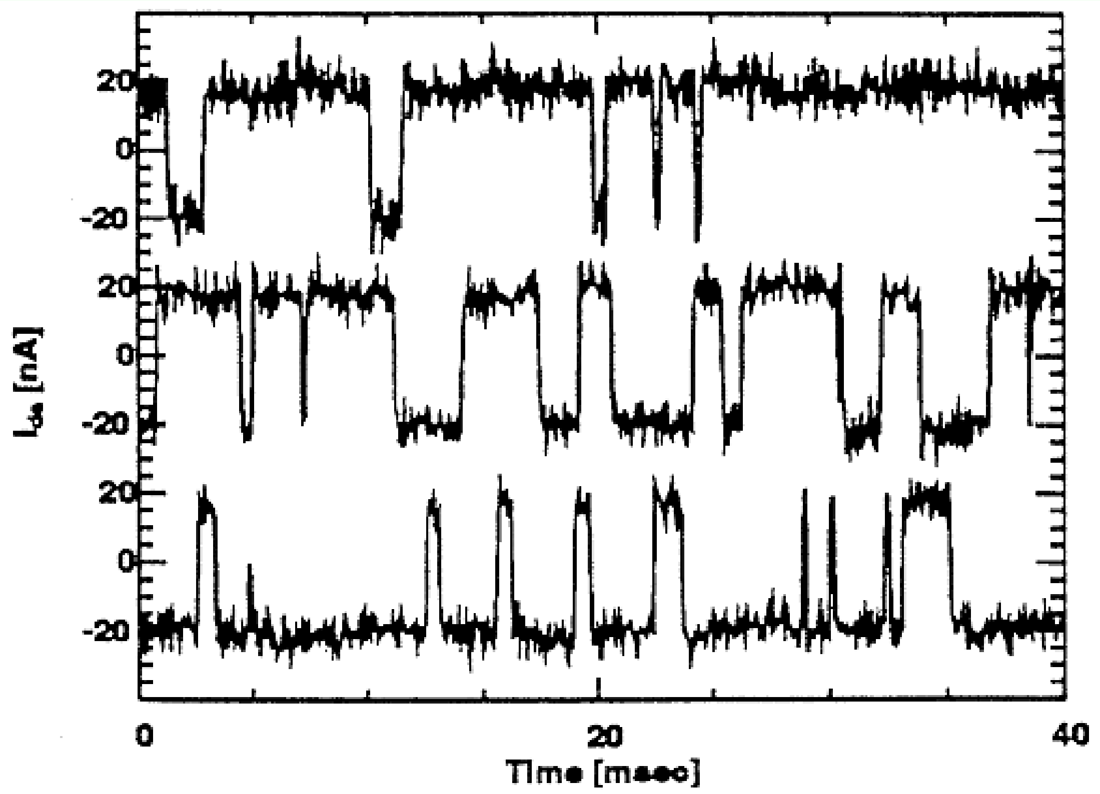


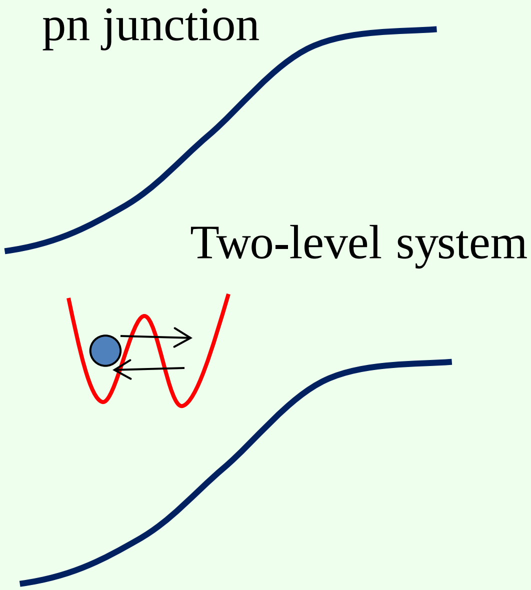


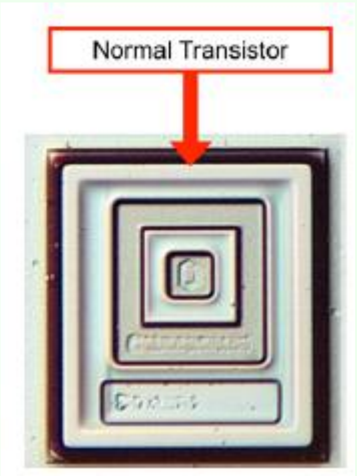


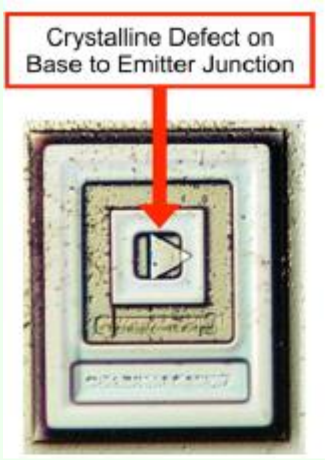
Popcorn noise

Popcorn noise, Burst noise

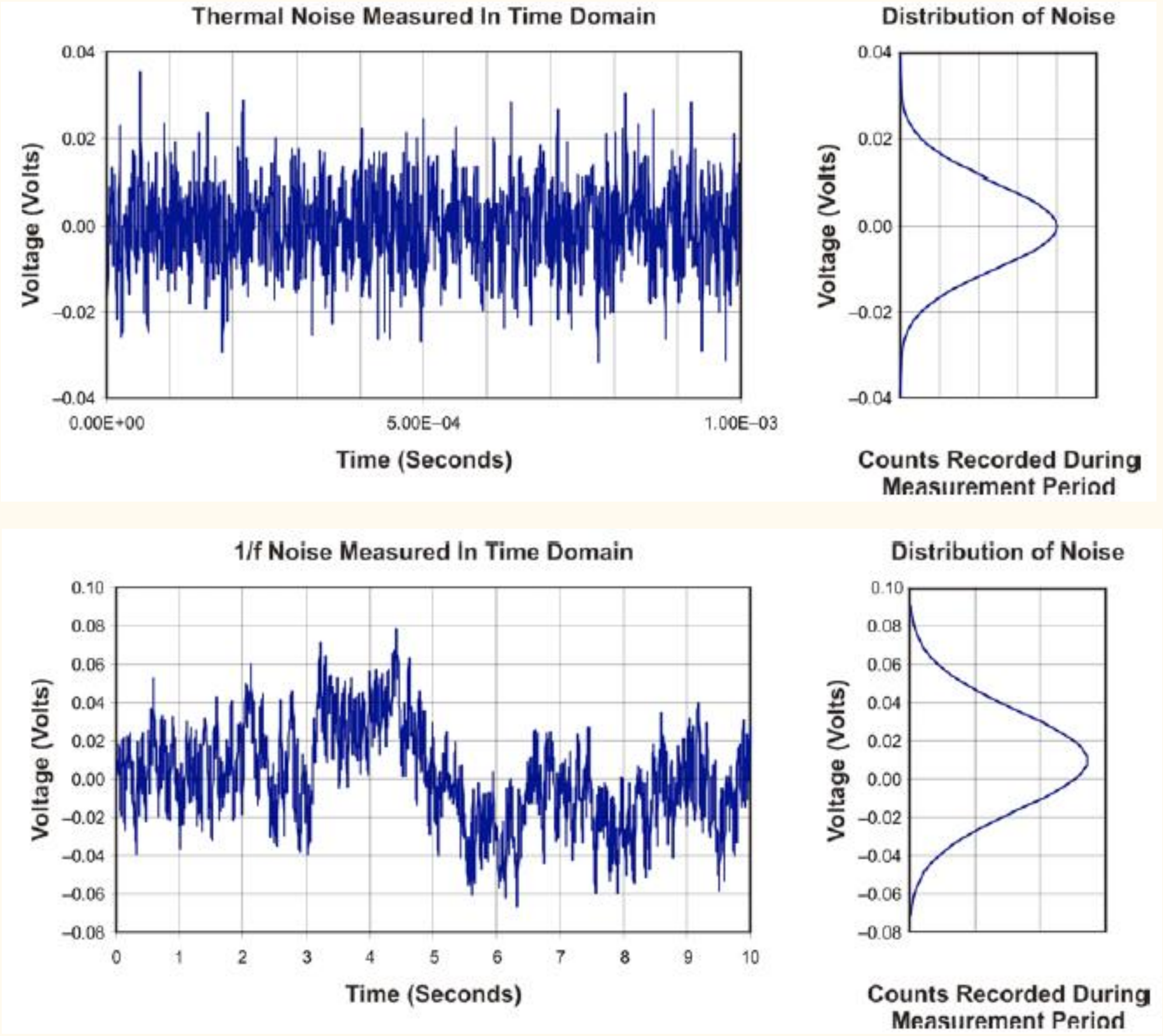




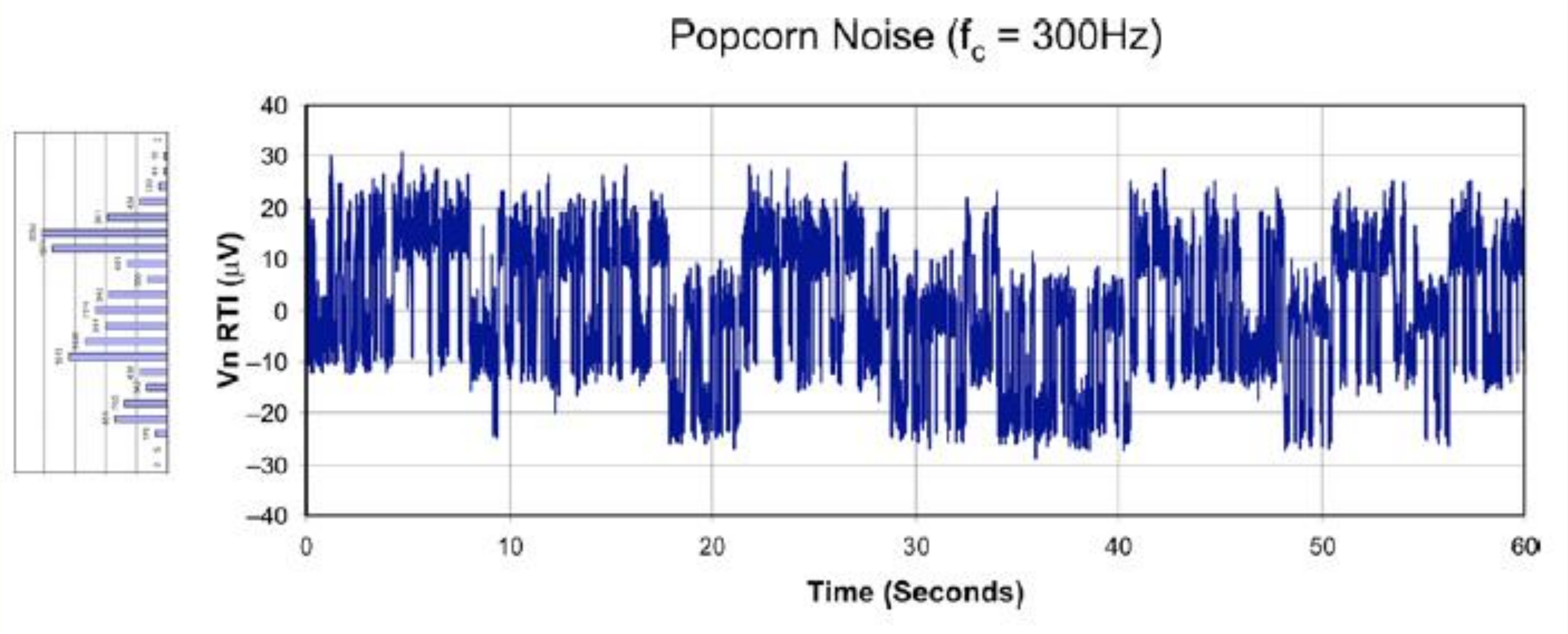




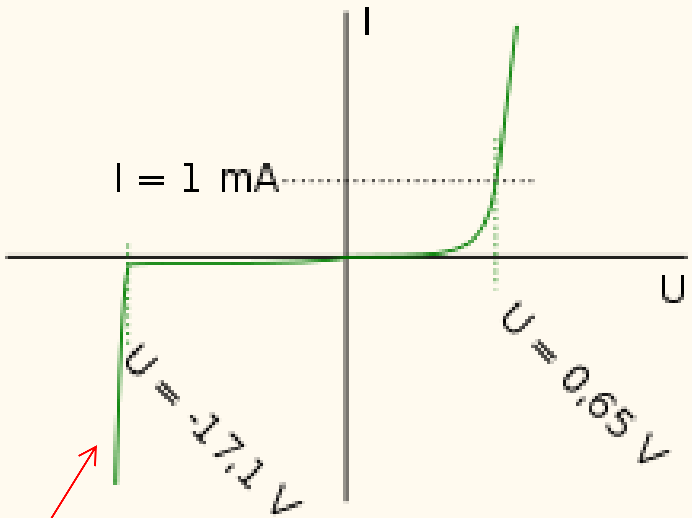
Amplitude distributions of random-type noises



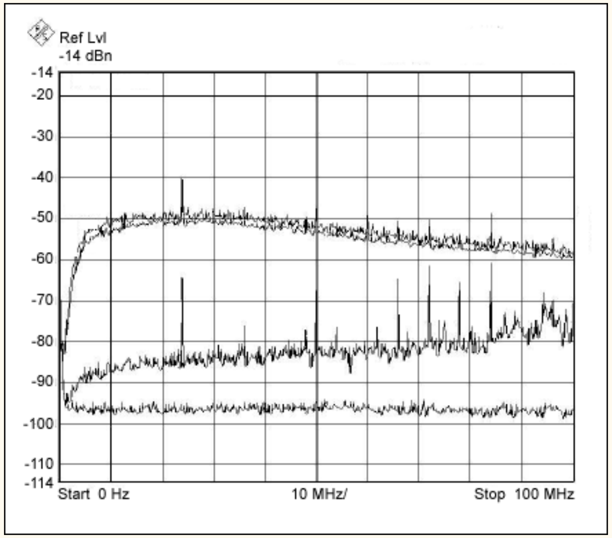
Amplitude distribution of popcorn noise



Avalanche noise

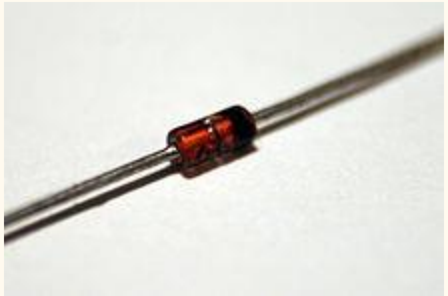


avalanche or Zener breakdown

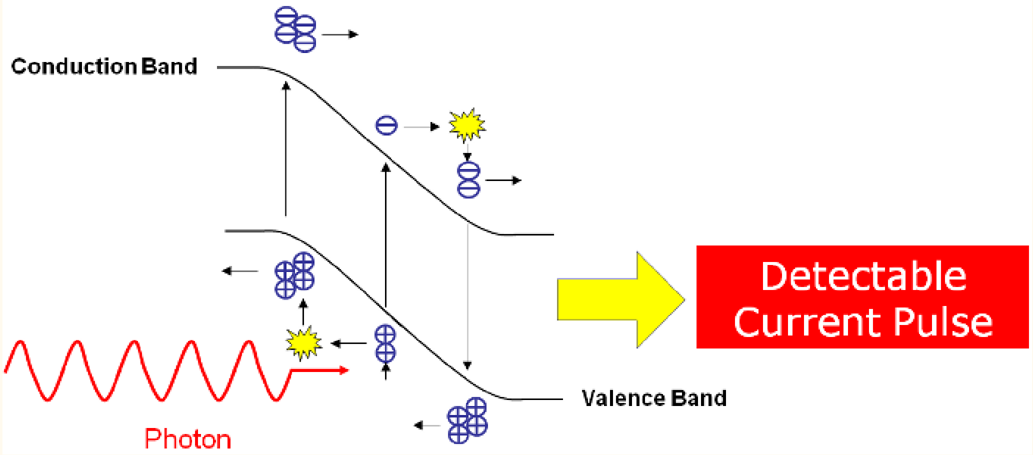


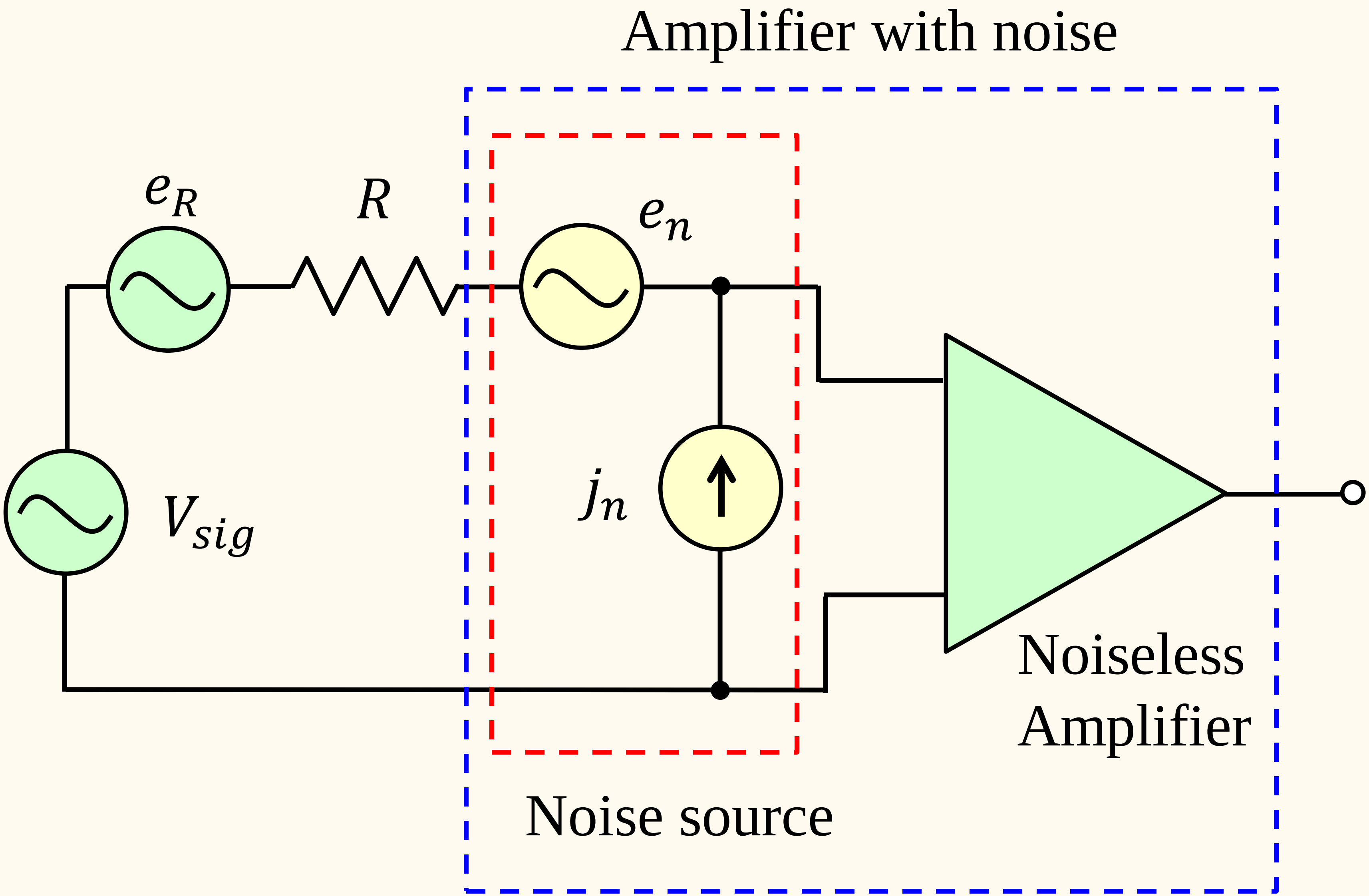
white noise

Avalanche Photo-Diode (APD)



Zener voltage standard diode




Amplifier with noise
e_R
R
e_n
j_n
V_{sig}
Noiseless
Amplifier
Noise source

6.2 Noises from Amplifiers

Amplifiers: the elements have characteristic noises,
power sources work as noise sources

⇒ Noiseless amplifier + Noise source = Amplifier with noise

Power gain G_p

$$e_{\mathrm{intotal}}^2 = j_n^2 R^2 + e_{\mathrm{R}}^2 + e_n^2 = e_{\mathrm{out}}^2 / G_{\mathrm{p}}$$

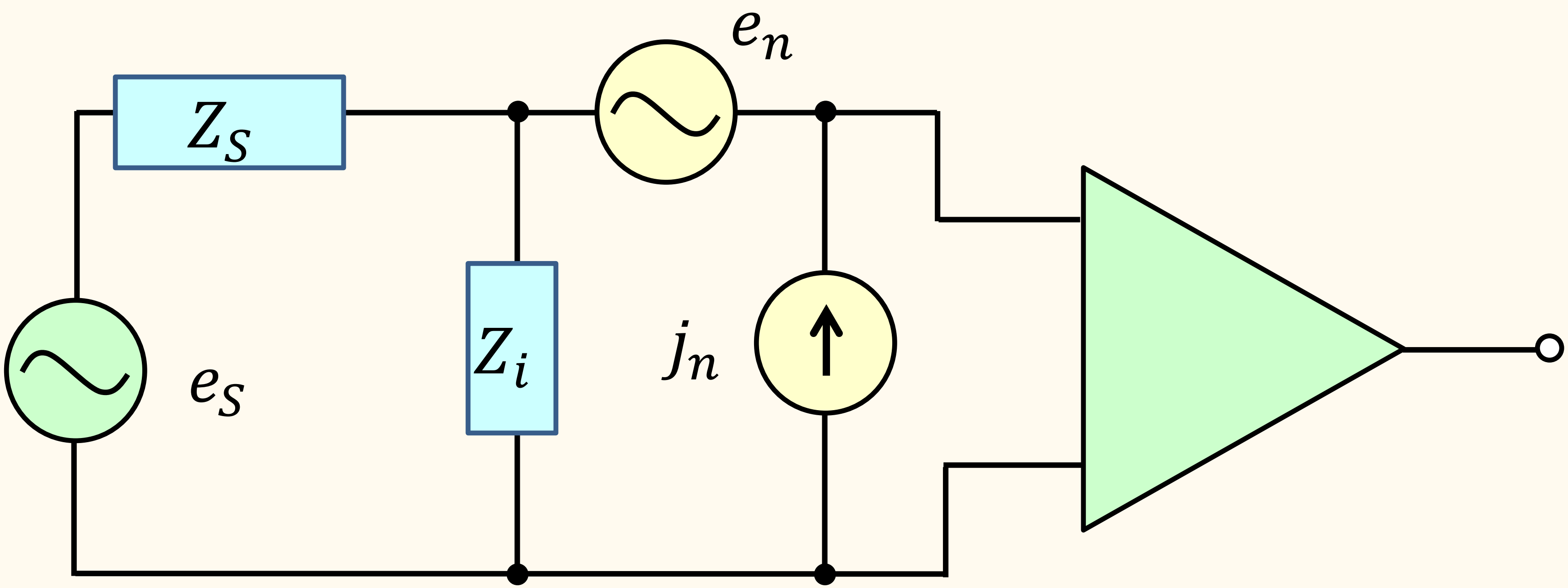
Signal to noise ratio: S/N ratio

Noise Figure: $\mathrm{NF} = 10\log_{10}\dfrac{(S/N)_{\mathrm{in}}}{(S/N)_{\mathrm{out}}} = 10\log_{10}\dfrac{S_{\mathrm{in}}N_{\mathrm{out}}}{S_{\mathrm{out}}N_{\mathrm{in}}}$

$$N_{\mathrm{out}} = G_{\mathrm{p}}\overline{e_{\mathrm{N}}^2}$$

$$\mathrm{NF} = 10\log_{10}\frac{S_{\mathrm{in}}G_{\mathrm{p}}\overline{e_{\mathrm{N}}^2}}{S_{\mathrm{in}}G_{\mathrm{p}}\overline{e_{\mathrm{R}}^2}} = 10\log_{10}\frac{\overline{e_{\mathrm{N}}^2}}{\overline{e_{\mathrm{R}}^2}} = 10\log_{10}\frac{\overline{e_n^2} + \overline{e_{\mathrm{R}}^2} + \overline{j_n^2}R^2}{\overline{e_{\mathrm{R}}^2}}$$

6.2.2 Noise impedance matching



6.2.2 Noise impedance matching

Noise temperature and matched source impedance

$$T_a = \frac{\sqrt{\overline{e_n^2}\ \overline{j_n^2}}}{2k_B}, \qquad R_{bs} = \sqrt{\frac{\overline{e_n^2}}{\overline{j_n^2}}}$$

Output noise temperature:

$$T_n = \left(1 + \frac{\mathrm{Re}(1/Z_i)}{\mathrm{Re}(1/Z_s)}\right) \frac{T_a}{2\mathrm{Re}Z} \left(\frac{|Z|^2}{R_{bs}} + R_{bs}\right), \qquad \frac{1}{Z} \equiv \frac{1}{Z_i} + \frac{1}{Z_s}$$

Minimize T_n: $\quad Z_i = \dfrac{1}{R_{bs}^{-1} - Z_s^{-1}}$ Noise matching condition

$$T_n = \left(1 + \frac{\mathrm{Re}(1/Z_i)}{\mathrm{Re}(1/Z_s)}\right) T_a$$

References

C. Kittel, "Elementary Statistical Physics", (Dover, 2004).

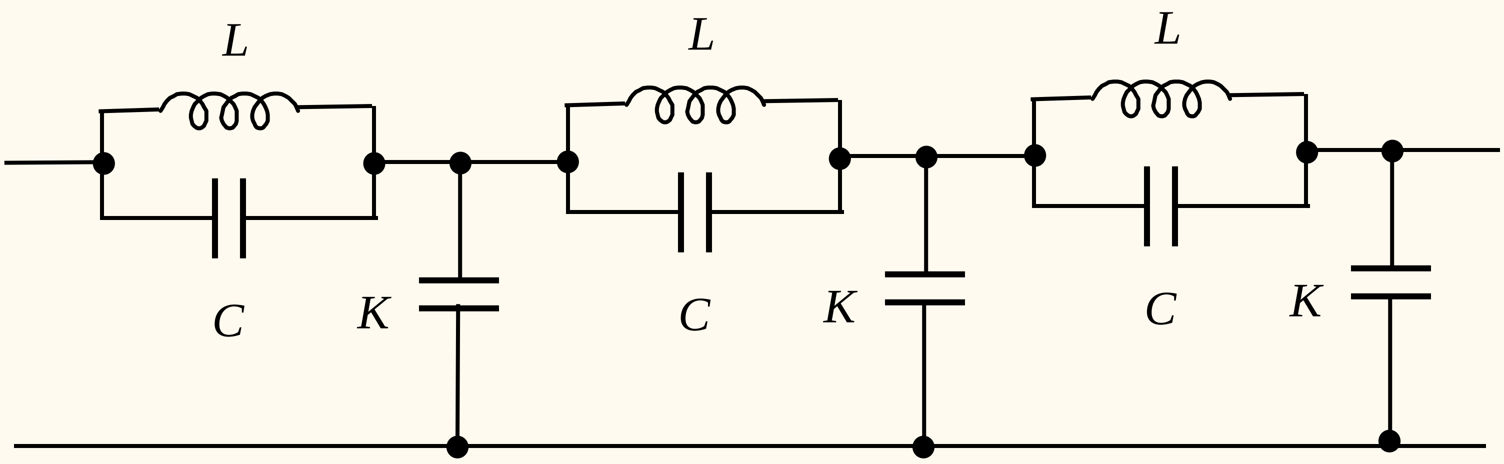
遠坂俊昭「計測のためのアナログ回路設計」(CQ出版社，1997).

Anton F. P. van Putten, "Electronic Measurement Systems", (IOP pub., 1996).

寺本英，広田良吾，武者利光，山口昌哉「無限・カオス・ゆらぎ」(培風館，1985).

Exercise E-1

Obtain the dispersion relation in the following transmission line.



Exercise E-2

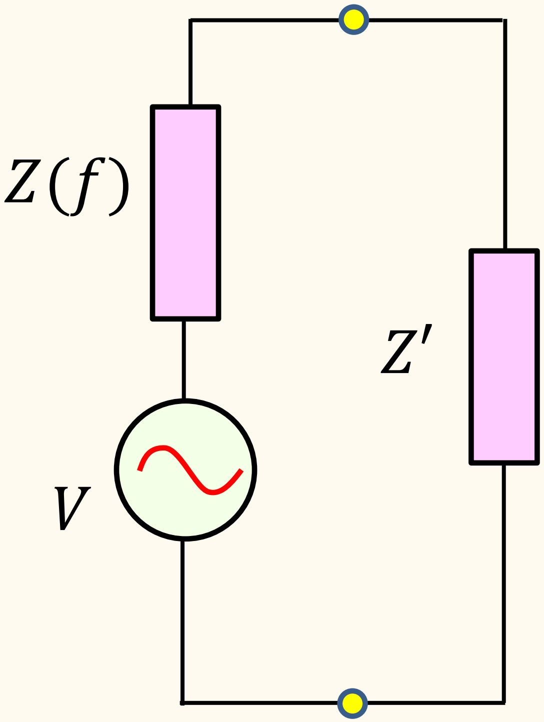
Show that the power spectrum $G(f)$ of voltage noise across the impedance

$$Z(f) = R(f) + iY(f)$$

is given as

$$G(f) = 4R(f)k_\mathrm{B}T.$$

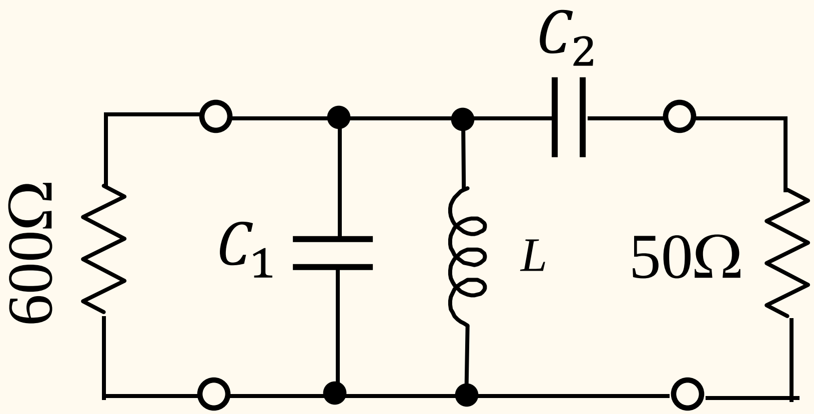
Assume that thermal noise energy per unit time is $k_B T \Delta f$.



(hint) From the above assumption we can skip the discussion on the mode energy in transmission line. Instead consider the case in the left figure, in which Z' is matched to Z as

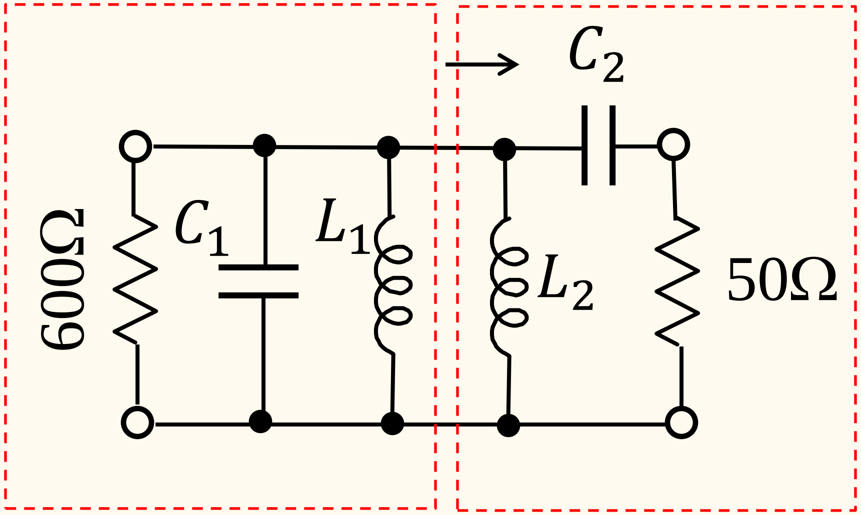
$$Z'(f) = Z^*(f) = R(f) - iY(f)$$

Exercise E-3



(a)

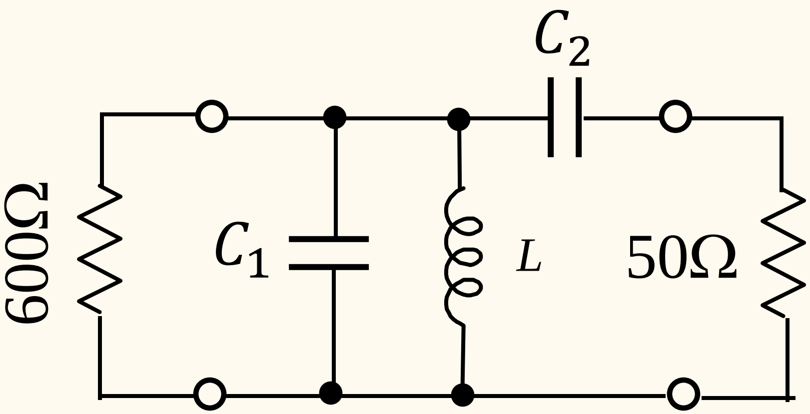
(b)



A preamplifier with FETs for an FM receiver has the output impedance of 600Ω. The FM receiver has the input impedance of 50Ω and we need to make impedance matching. The central frequency is 85MHz, the effective with of amplification is 10MHz. Obtain C_1, C_2, L in the matching circuit with 3 digits significant figures.

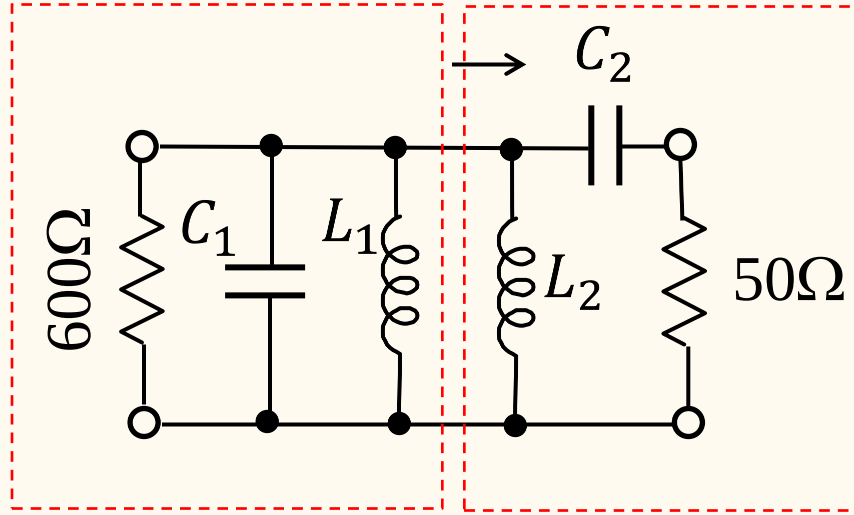
(hint) Express L with a parallel of L_1 and L_2 as shown in (b). The left resonance circuit should be tuned to 85MHz, width10MHz. Then the left and the right circuit should be impedance matched.

Exercise E-3



(a)

(b)



FM受信機のプリアンプをFETで作ったところ，出力インピーダンスが600Ωになった．受信機の入力インピーダンスは50Ωなので，インピーダンスマッチを取る必要がある．中心周波数を85MHz，有効周波数幅を10MHz，として(a)のような回路でマッチを取ると，回路定数C_1, C_2, Lはどうなるか．有効数字3桁で答えよ．

(ヒント) (b)のようにインダクタンスを２つに分割し，左の共鳴回路で85MHz，10MHz幅に同調させる．この後，左右のインピーダンスが一致するように定数を求める．