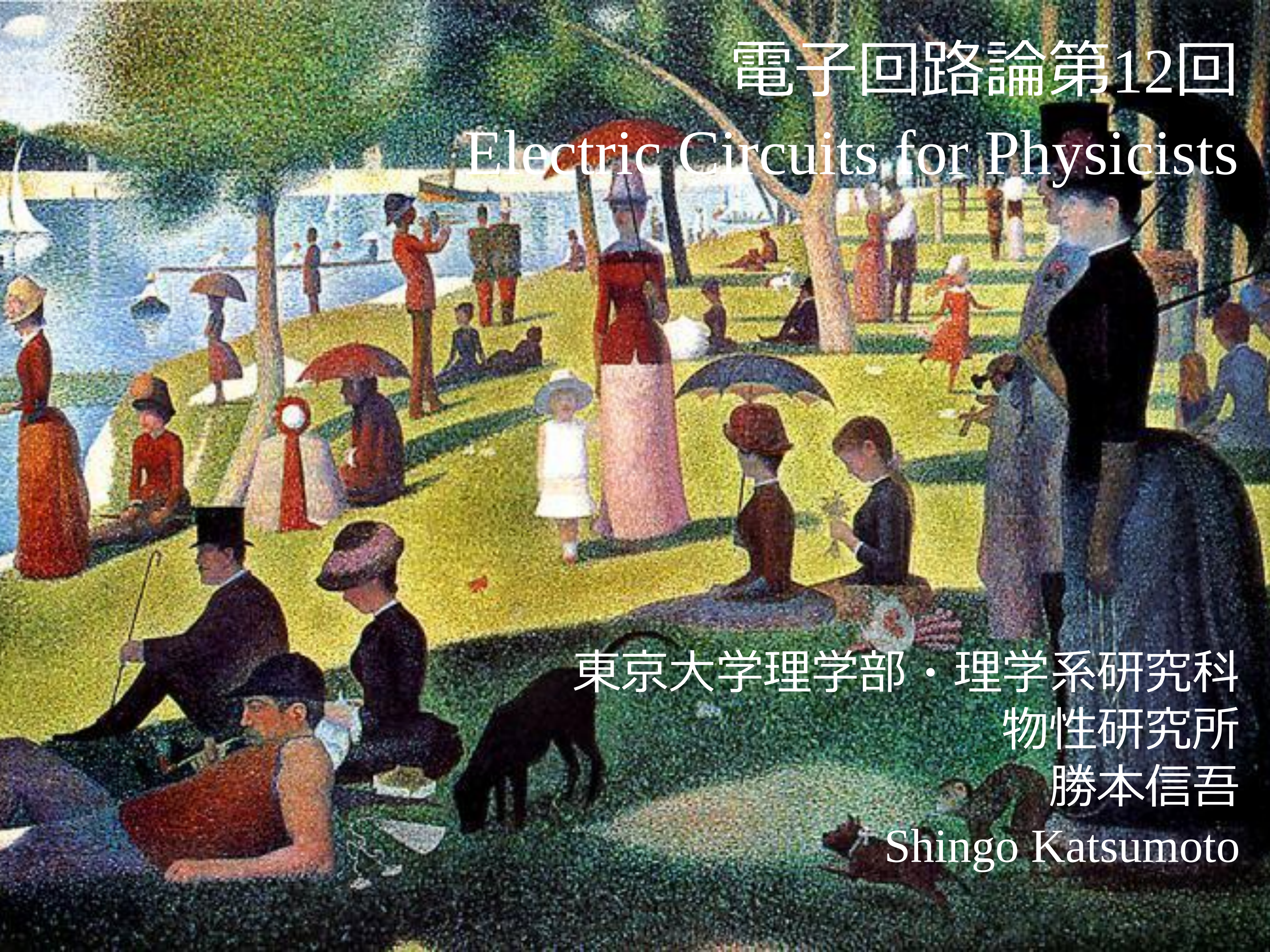


電子回路論第12回
Electric Circuits for Physicists

東京大学理学部・理学系研究科
物性研究所
勝本信吾
Shingo Katsumoto


Slopestyle

Outline

6.4 Discrete signal

- 6.4.1 Sampling theorem
- 6.4.2 Pulse amplitude modulation (PAM)
- 6.4.3 Discrete Fourier transform
- 6.4.4 z-transform
- 6.4.5 Transfer function of discrete time signal

Ch.7 Digital signals and circuits

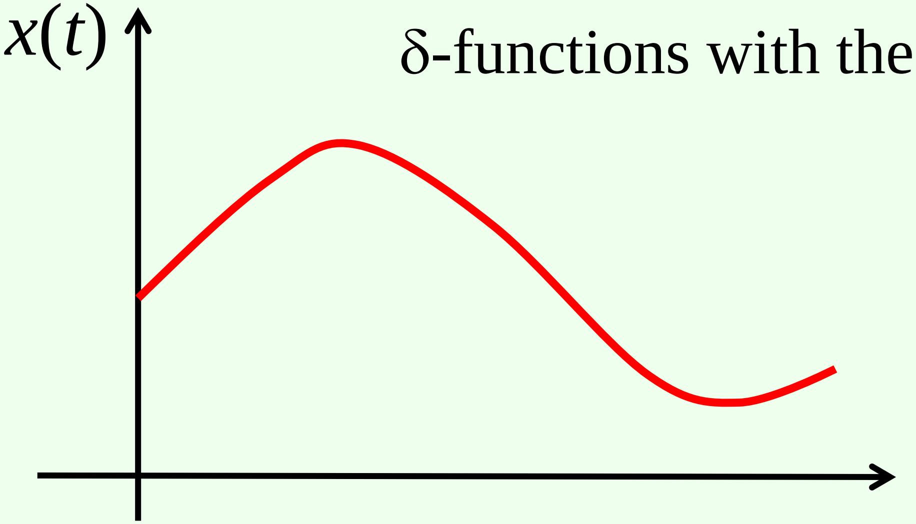
- 7.2 Logic gates
- 7.3 Implementation of logic gates
- 7.4 Circuit implementation and simplification of logic operation

6.4 Discrete signal

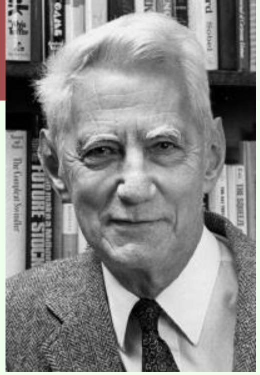
6.4.1 Sampling theorem

Sampled signal $\tilde{x}(t) = x(t)\underline{\delta_\tau(t)}$

δ-functions with the period τ

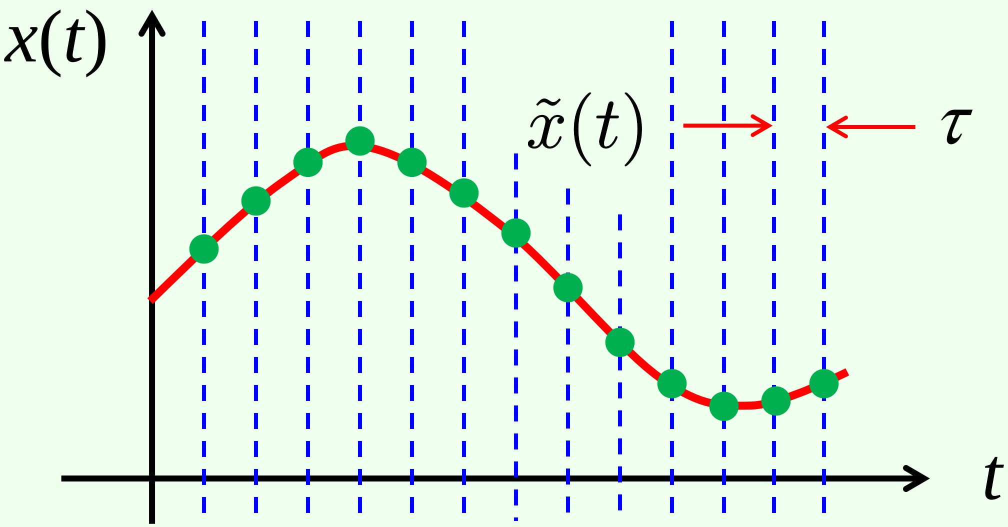


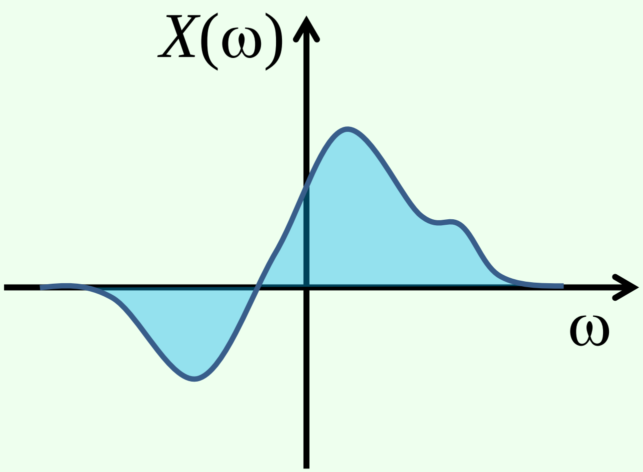
Isao Someya 1915-2007



Claude Shannon 1916-2001

1928 H. Nyquist
1949 C. Shannon
染谷勲





6.4.1 Sampling theorem

$$\delta_\tau(t) = \sum_{j=-\infty}^{\infty} \delta(t - j\tau) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{\tau}\int_{-\pi/\tau}^{\pi/\tau} \delta(s)ds\right] \exp\left(-in\frac{2\pi}{\tau}t\right)$$

$$= \frac{1}{\tau}\sum_{n=-\infty}^{\infty} \exp\left(-in\frac{2\pi}{\tau}t\right)$$

$$\mathscr{F}\{\delta_\tau(t)\} = \int_{-\infty}^{\infty}\left[\frac{1}{\tau}\sum_{-\infty}^{\infty} e^{-in(2\pi/\tau)t}\right] e^{i\omega t}dt = \frac{1}{\tau}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[i\left(\omega - n\frac{2\pi}{\tau}\right)t\right]dt$$

$$= \frac{2\pi}{\tau}\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{\tau}\right) = \frac{2\pi}{\tau}\delta_{2\pi/\tau}(\omega)$$

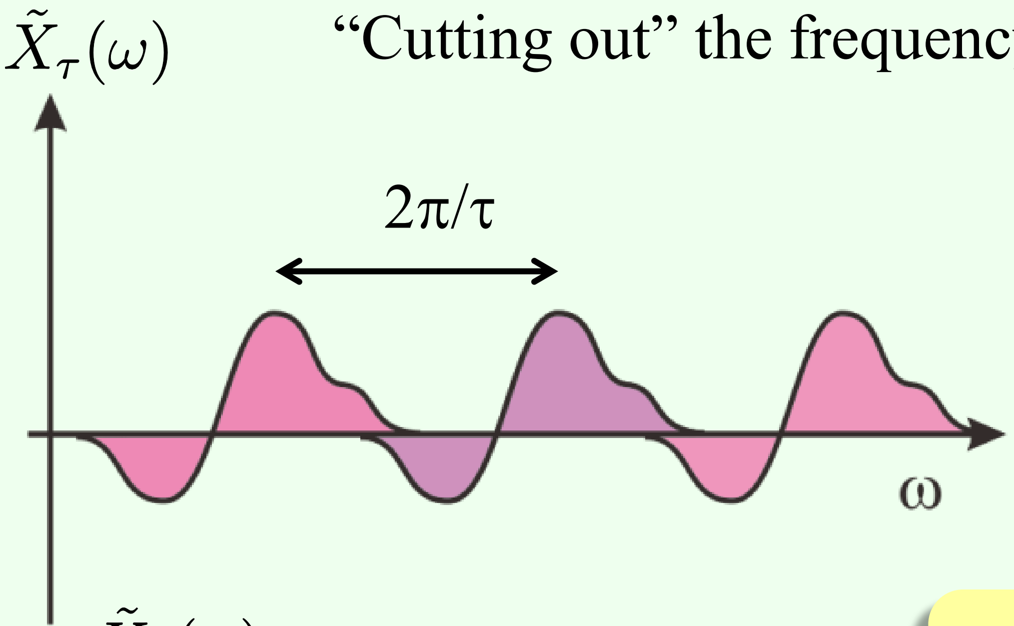
$$\mathscr{F}\{x(t)\} = X(\omega),\ \mathscr{F}\{\tilde{x}_\tau(t)\} = \tilde{X}_\tau(\omega)$$

$$\tilde{X}_\tau(\omega) = \frac{1}{2\pi}X(\omega) * \frac{2\pi}{\tau}\delta_{2\pi/\tau}(\omega) = \frac{1}{\tau}X(\omega) * \sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{\tau}\right)$$

$$= \frac{1}{\tau}\int_{-\infty}^{\infty} X(\omega')\left\{\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\frac{2\pi}{\tau} - \omega'\right)\right\}d\omega' = \frac{1}{\tau}\sum_{n=-\infty}^{\infty} X\left(\omega - n\frac{2\pi}{\tau}\right)$$

6.4.1 Sampling theorem

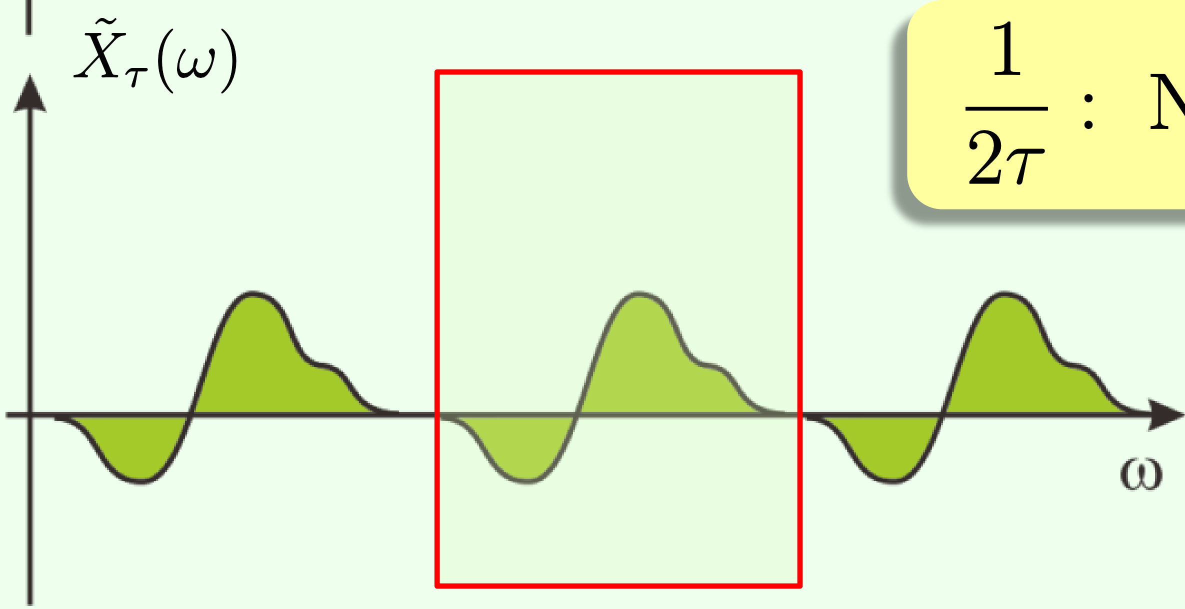
“Cutting out” the frequency spectrum



ω_h: Highest frequency in $\tilde{X}_\tau(\omega)$

$$\frac{2\pi}{\tau} > 2\omega_h, \quad \tau < \frac{\pi}{\omega_h}$$

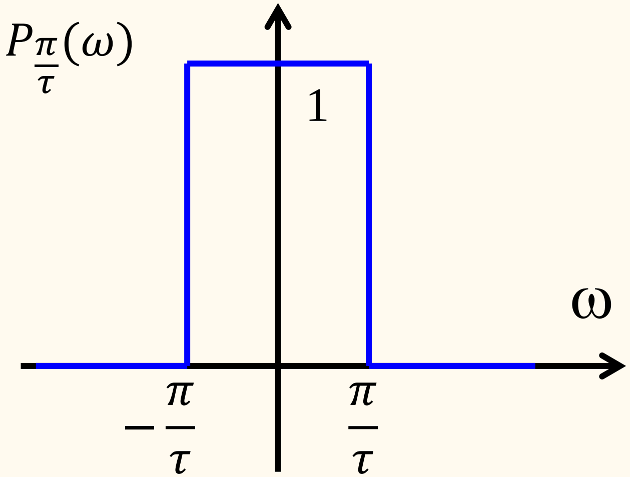
$$\frac{1}{2\tau} : \text{ Nyquist frequency}$$

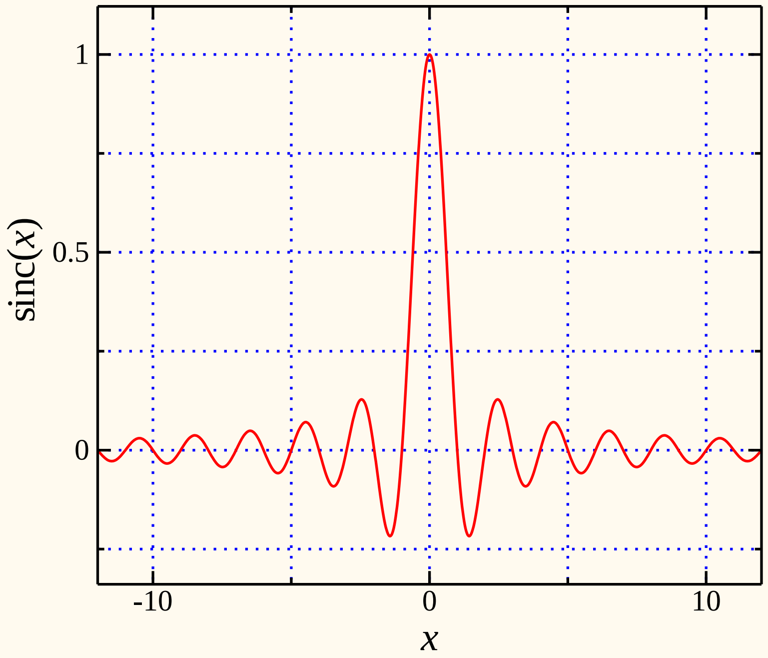


6.4.1 Sampling theorem: Reconstructing signal

$$P_{\pi/\tau}(\omega) = \begin{cases} 1 & |\omega| \le \dfrac{\pi}{\tau}, \\ 0 & |\omega| > \dfrac{\pi}{\tau} \end{cases}$$

$$x(t) = \mathscr{F}^{-1}\{\tau P_{\pi/\tau}(\omega)\tilde{X}_\tau(\omega)\}$$

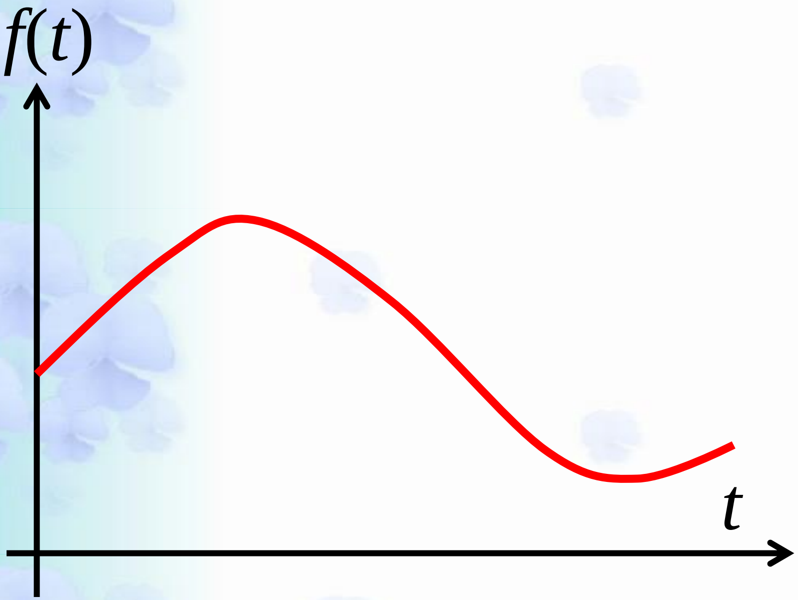


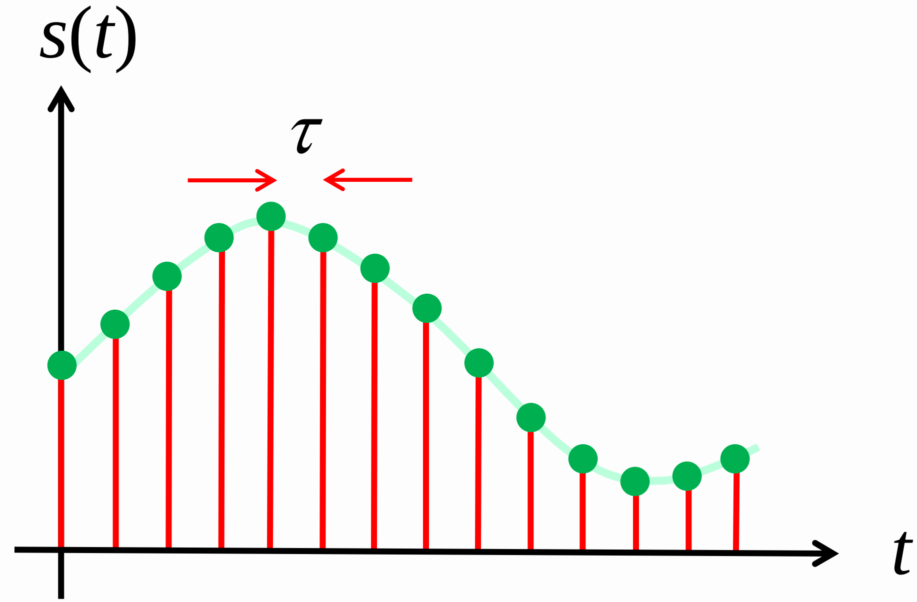


$$x(t) = \tau\frac{1}{\tau}\mathrm{sinc}\left(\frac{t}{\tau}\right) * \tilde{x}_\tau(t) = \mathrm{sinc}\left(\frac{t}{\tau}\right) * \sum_{n=-\infty}^{\infty} x(t)\delta(t - n\tau)$$

$$= \int_{-\infty}^{\infty} \mathrm{sinc}\left(\frac{s}{\tau}\right) \sum_{n=-\infty}^{\infty} x(t-s)\delta(t - n\tau - s)ds = \sum_{n=-\infty}^{\infty} \mathrm{sinc}\left(\frac{t-n\tau}{\tau}\right) x(n\tau)$$

6.4.2 Pulse amplitude modulation (PAM)





Carrier: $\delta_\tau(t)$ $$s(t) = f(t)\delta_\tau(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\tau)$$

Demodulation = Reconstruction of continuous signal from sampled data.

$$f(t) = \mathscr{F}^{-1}\{P_{\pi/\tau}(\omega)\mathscr{F}\{s(t)\}\}$$

Demodulation of PAM and a trick in the sampling theorem

In the sampling theorem, though we only have discrete-time data, we can reconstruct complete original signal.

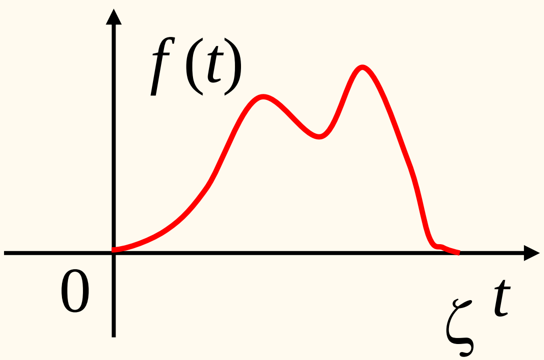
↑

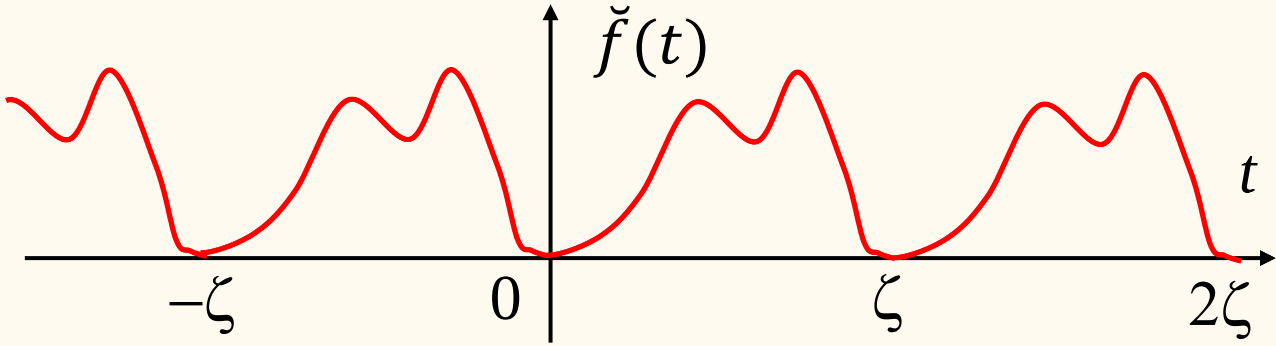
Assumption: we have data in infinite period $[-\infty, +\infty]$.

However in actual situations we can never have such data.

Need to consider handling data in a finite period.

6.4.3 Discrete Fourier transform





Assumption:

$F(\omega) = \mathscr{F}\{f(t)\}$, not zero in $\omega \in \left(-\frac{\pi}{\tau}, \frac{\pi}{\tau}\right)$

$$N = \frac{\zeta}{\tau} \in \mathbb{N}$$

can be assumed without loosing generality

$$\breve{f}(t) = \sum_{n=-\infty}^{\infty} f(t - n\zeta), \quad \breve{F}(\omega) = \sum_{n=-\infty}^{\infty} F\left(\omega + n\frac{2\pi}{\zeta}\right)$$

$$\left(\breve{f}(t) = (f * \delta_\zeta)(t) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi)\delta(t - n\zeta - \xi)d\xi\right)$$

Fourier expansion: $$\breve{f}(t) = \frac{1}{\zeta} \sum_{n=-\infty}^{\infty} F\left(n\frac{2\pi}{\zeta}\right) \exp\left(2n\pi i\frac{t}{\zeta}\right)$$

6.4.3 Discrete Fourier transform

$$n = l + mN \qquad \sum_{n=-\infty}^{\infty} \to \sum_{l=0}^{N-1} \sum_{m=-\infty}^{\infty}$$

Discreteness:

$$t = j\tau \quad j \in \mathbb{Z}$$

$$\breve{f}(j\tau) = \frac{1}{\zeta} \sum_{l=0}^{N-1} \sum_{m=-\infty}^{\infty} F\left[(l+mN)\frac{2\pi}{\zeta}\right] \exp\left[(l+mN)2\pi i \frac{j\tau}{\zeta}\right]$$

$$= \frac{1}{N\tau} \sum_{l=0}^{N-1} \sum_{m=-\infty}^{\infty} F\left(\frac{2\pi l}{\zeta} + m\frac{2\pi}{\tau}\right) \exp\left(2\pi i \frac{lj}{N}\right)$$

$$= \frac{1}{N\tau} \sum_{l=0}^{N-1} \breve{F}\left(l\frac{2\pi}{\zeta}\right) \exp\left(2\pi i \frac{lj}{N}\right)$$

Twiddle factor: $W_N \equiv \exp\left(-i\frac{2\pi}{N}\right)$

$$\eta \equiv \frac{2\pi}{\zeta} \qquad \breve{f}(j\tau) = \frac{1}{N\tau} \sum_{l=0}^{N-1} \breve{F}(l\eta) W_N^{-lj}$$

6.4.3 Discrete Fourier transform

Twiddle factor:

$$\forall n, m \in \mathbb{Z} \quad W_N^{n+mN} = W_N^n,$$

$$\frac{1}{N}\sum_{n=0}^{N-1} W_N^{nm} = \begin{cases} 1 & \text{for} \quad m = 0, \\ 0 & \text{for} \quad m \neq 0. \end{cases}$$

$$\tau \sum_{j=0}^{N-1} \breve{f}(j\tau) W_N^{mj} = \sum_{j=0}^{N-1}\left[\frac{1}{N}\sum_{l=0}^{N-1} \breve{F}(l\eta) W_N^{(m-l)j}\right] = \breve{F}(m\eta)$$

$$f_n \equiv \breve{f}(n\tau), \quad F_k \equiv \frac{1}{\tau}\breve{F}(k\eta)$$

Discrete Fourier transform:
(DFT)

$$F_k = \sum_{n=0}^{N-1} f_n W_N^{kn},$$

$$f_n = \frac{1}{N}\sum_{k=0}^{N-1} F_k W_N^{-kn}.$$

6.4.3 Discrete Fourier transform

$$\boldsymbol{F} = {}^t\{F_i\},\ \boldsymbol{W} = \{W_N^{ij}\},\ \boldsymbol{f} = {}^t\{f_i\}$$

$$\boldsymbol{F} = \boldsymbol{W}\boldsymbol{f}, \quad \boldsymbol{f} = \frac{1}{N}\boldsymbol{W}^*\boldsymbol{F}$$

$${}^t\boldsymbol{W}^*\boldsymbol{W} = N\boldsymbol{I}_N \quad i.e., \ \frac{1}{\sqrt{N}}\boldsymbol{W} \ \text{: unitary}$$

Discrete Laplace transform: z-transform

$$\tilde{f}_\tau(t) = \sum_{n=0}^{\infty} f(n\tau)\delta(t - n\tau) \quad (t \geq 0)$$

$$\mathscr{L}\{\tilde{f}_\tau(t)\}(s) = \mathscr{L}\left\{\sum_{n=0}^{\infty} f(n\tau)\delta(t - n\tau)\right\}$$

$$= \sum_{n=0}^{\infty} f(n\tau)\mathscr{L}\{\delta(t - n\tau)\} = \sum_{n=0}^{\infty} f(n\tau)\exp(-sn\tau)$$

$$z = \exp(s\tau),\ f_n = f(n\tau),\ F(z) = \mathscr{L}\{\tilde{f}_\tau(t)\}$$

$$F(z) = \sum_{n=0}^{\infty} f_n z^{-n} = \mathscr{Z}[\tilde{f}_\tau(t)]$$

one-sided z-transform

6.4.4 z-transform

f_n	$F(z)$	conversion area
$\delta(n)$	1	z-plane
1	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
n	$\dfrac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
n^k	$\left(-z\dfrac{d}{dz}\right)^k \dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
a^n	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$\sin(n\omega\tau)$	$\dfrac{\sin(\omega\tau)z^{-1}}{1-2\cos(\omega\tau)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$e^{-n\alpha\tau}\cos(n\omega\tau)$	$\dfrac{1-e^{-\alpha\tau}\cos(\omega\tau)z^{-1}}{1-2e^{-\alpha\tau}\cos(\omega\tau)z^{-1}+e^{-2\alpha\tau}z^{-2}}$	$\lvert z\rvert > e^{-\alpha\tau}$

6.4.4 z-transform

Property	Signal	z-transform
linearity	$af_n + bg_n$	$aF(z) + bG(z)$
z-domain scaling	$f_{\alpha n}$	$F(z^{1/\alpha})$
time shift	f_{n+k}	$z^k \left[F(z) - \sum_{l=0}^{k-1} f(l) z^l \right]$
time shift II	f_{n-k}	$z^{-k} F(z)$
scaling	$e^{\mp \alpha n} f_n$	$F(e^{\pm \alpha} z)$
scaling II	$a^n x_n$	$F(a^{-1} z)$
product with index	$n f_n$	$-z \frac{d}{dz} F(z)$
differentiation	$n^k f_n$	$\left(-z \frac{d}{dz} \right)^n F(z)$
integration	$\frac{f_n}{n+a}$	$z^a \int_z^\infty \xi^{-a+1} F(\xi) d\xi$
convolution	$f_n * g_n$	$F(z) \cdot G(z)$
product	$f_n \cdot g_n$	$\frac{1}{2\pi i} \oint_c F(\xi) G\left(\frac{z}{\xi} \right) \xi^{-1} d\xi$

6.4.5 Transfer function for discrete time signal

$$\tilde{f}_\tau(t) = f(t)\delta_\tau(t) = \sum_{k=-\infty}^{\infty} f_k \delta(t - k\tau)$$

h_n: (impulse) response to $\delta(n\tau)$, response to discrete signal $f_n = f\,(n\tau)$

$$g_n = \mathscr{R}\{\tilde{f}_\tau(n\tau)\} = \mathscr{R}\left\{ \sum_{k'=-\infty}^{\infty} f(k'\tau)\delta[(n-k')\tau] \right\}$$

$$= \sum_{k'=-\infty}^{\infty} f_{k'} h_{n-k'} = \sum_{k=-\infty}^{\infty} h_k f_{n-k}$$

$$G(z) = \mathscr{Z}[g_n] = \mathscr{Z}\left[\sum_{k=0}^{\infty} h_k f_{n-k} \right] = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} h_k f_{n-k} \right) z^{-n}$$

$$= \sum_{k=0}^{\infty} h_k \sum_{n=0}^{\infty} f_{n-k} z^{-n} = \sum_{k=0}^{\infty} h_k z^{-k} F(z)$$

$$H(z) = \mathscr{Z}[h_n] = \sum_{k=0}^{\infty} h_k z^{-k}$$: Transfer function

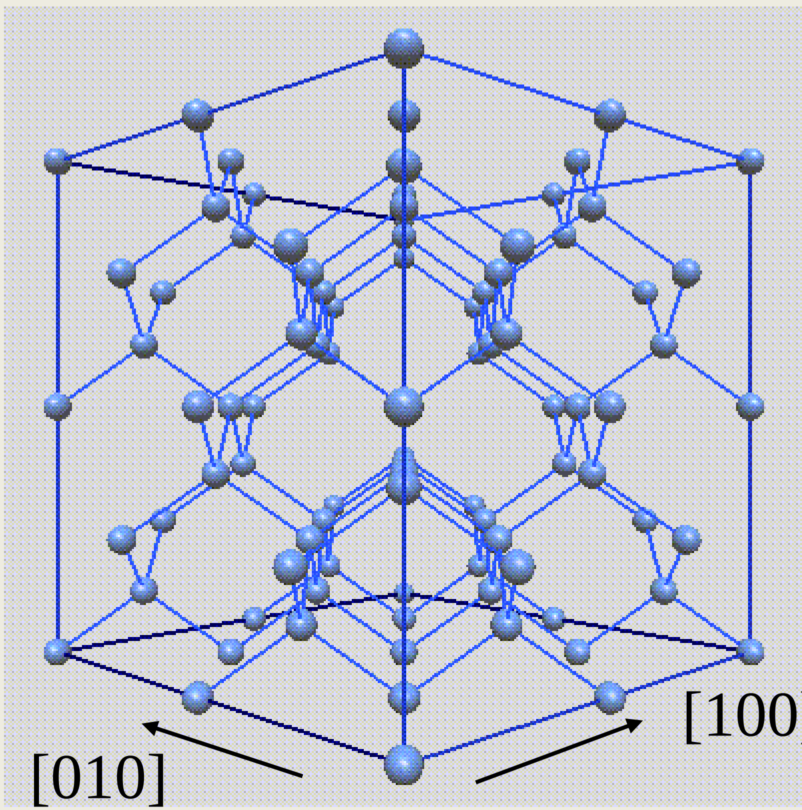
$$G(z) = H(z)F(z)$$

Crystal lattice and X-ray diffraction

Max von Laue
1879-1960

Laue pattern

Diamond lattice



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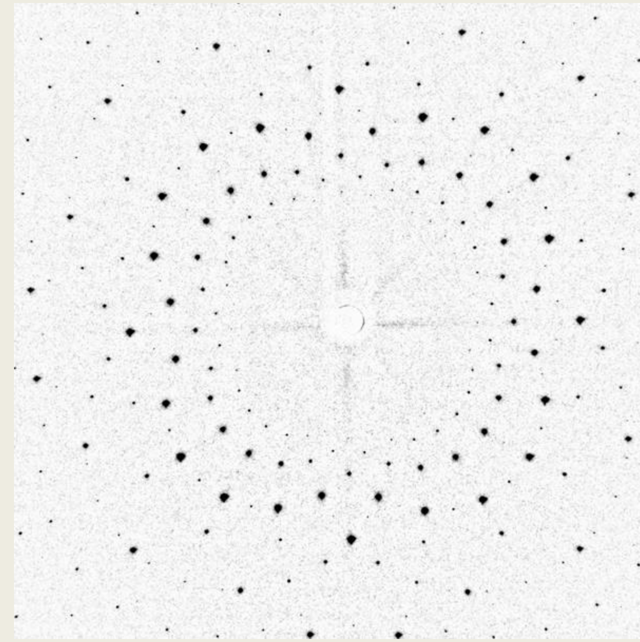

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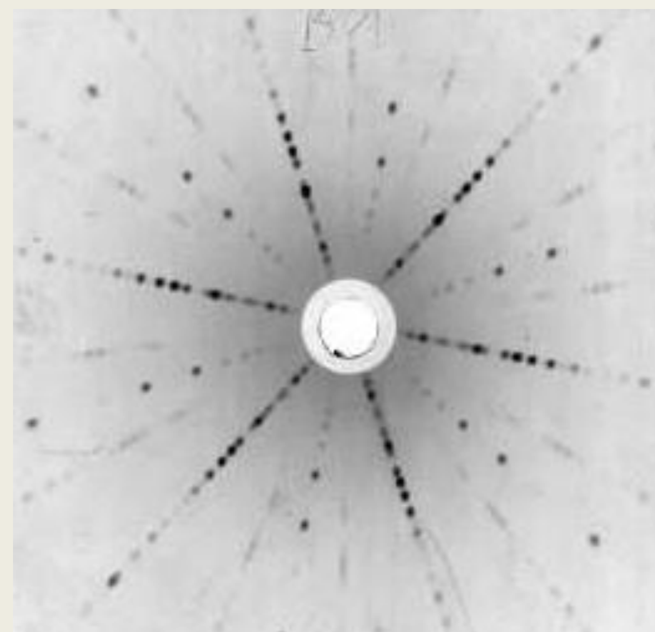
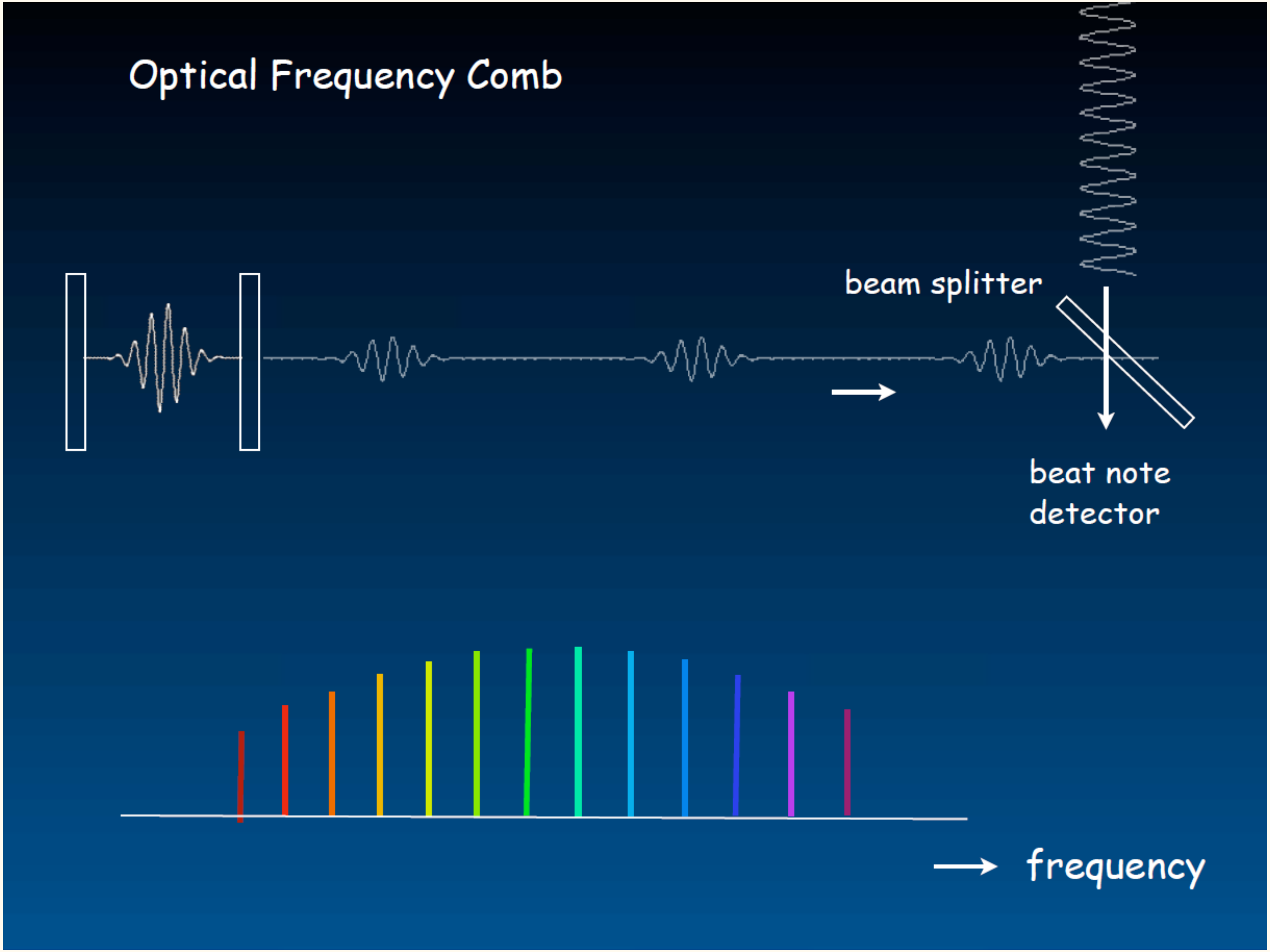
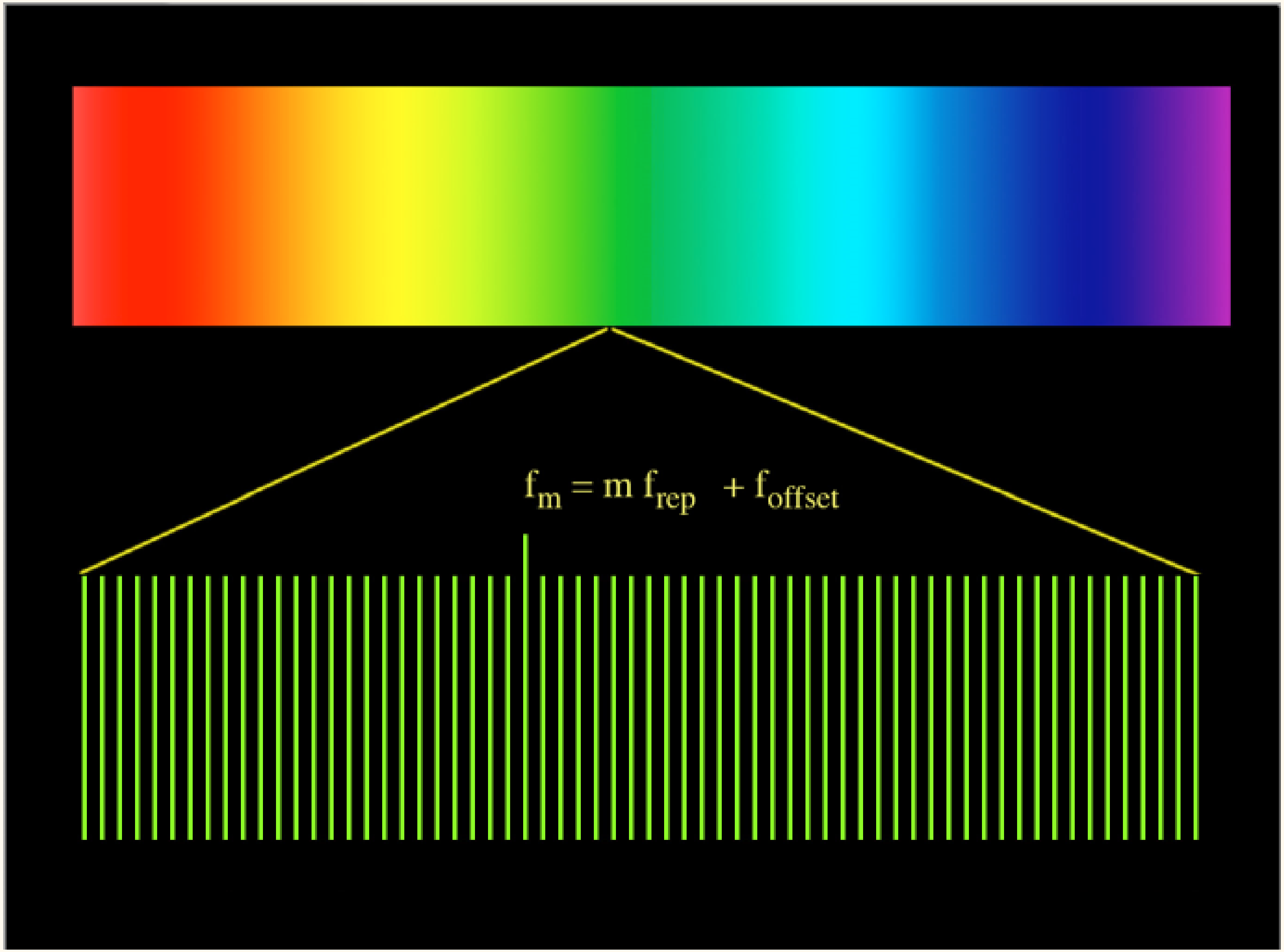


Optical Frequency Comb

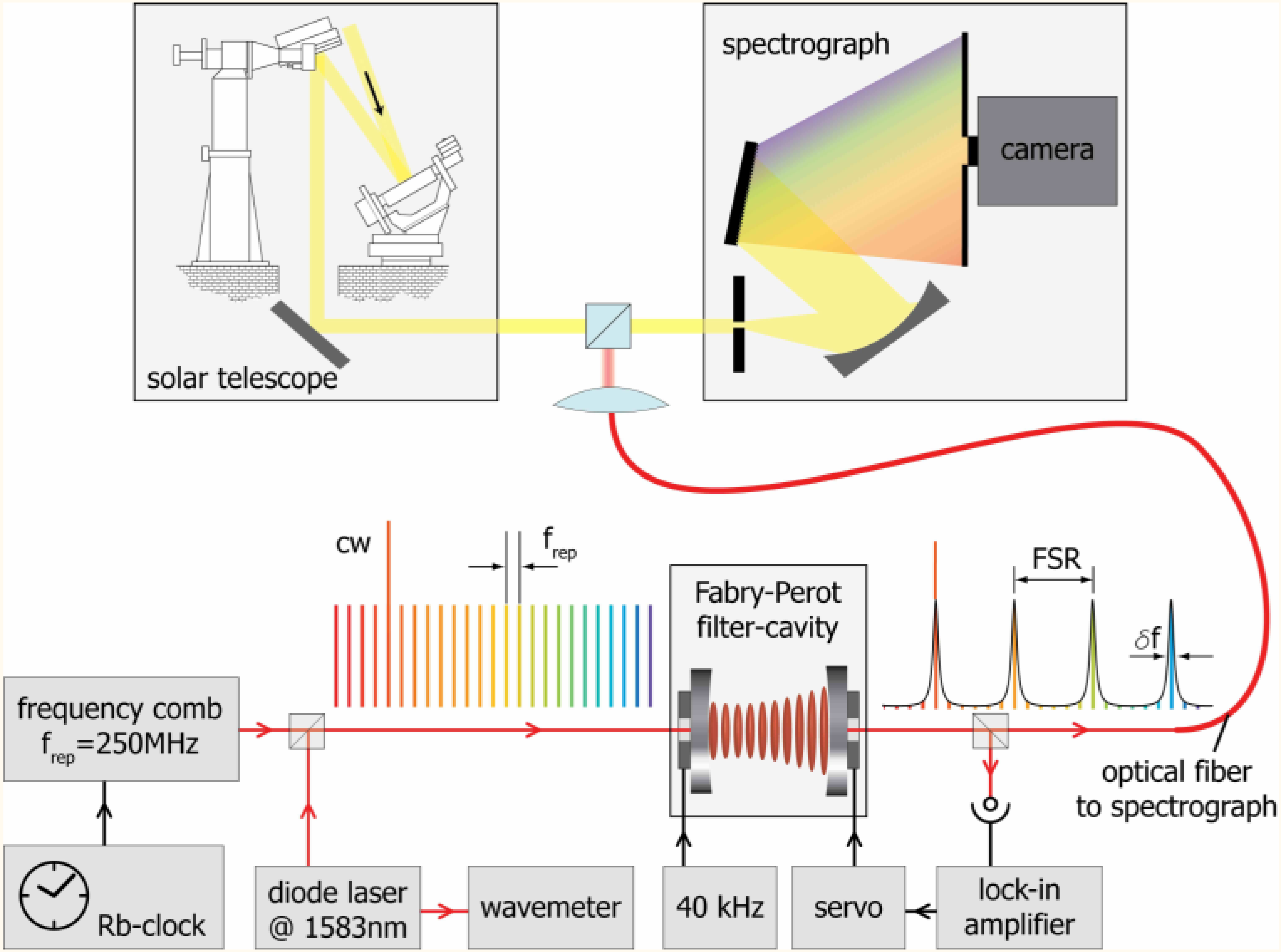


Frequency Comb



Theodor Hänsch, Max-Planck Institute, Science 2008

Measurement of the Doppler effect in cosmic expansion

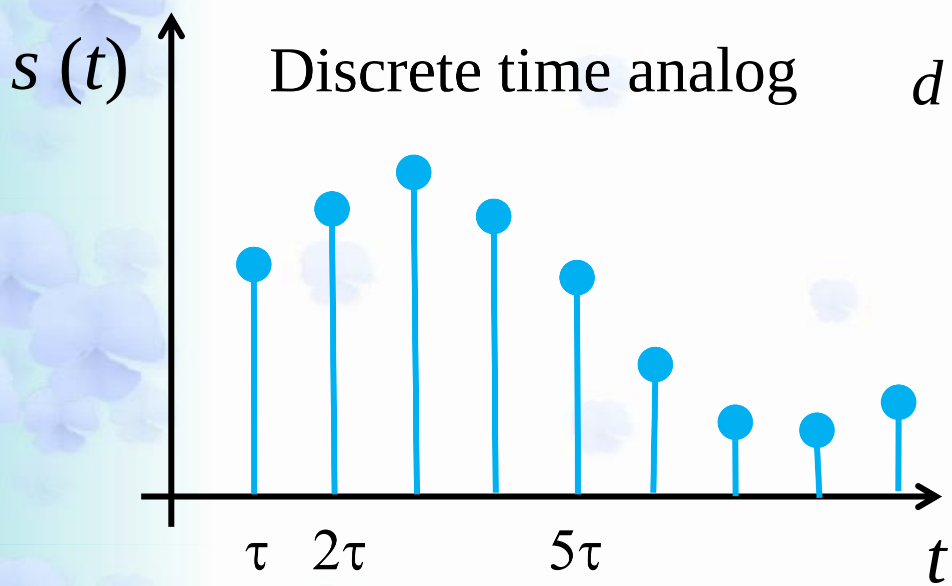


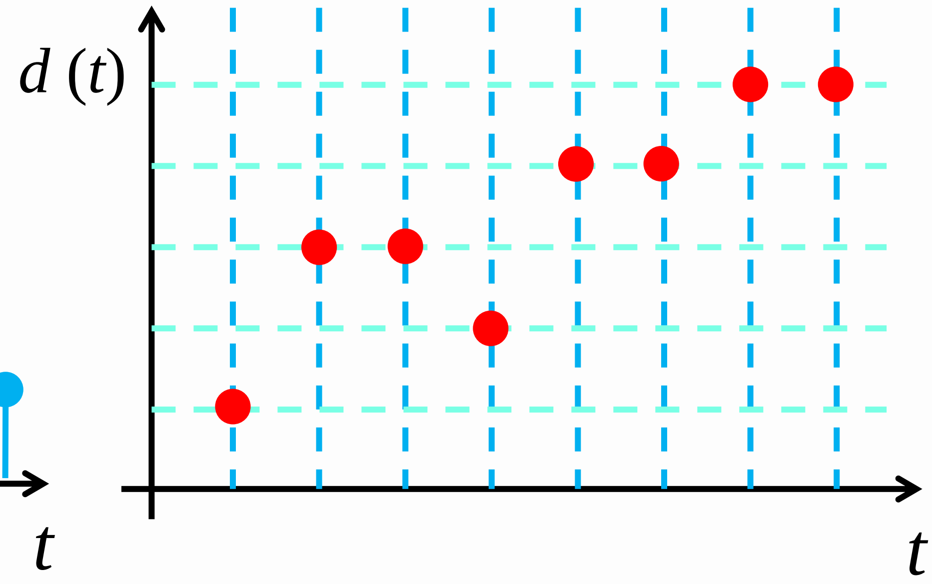
Byzantine mosaic

Chapter 7

Digital signal and circuits

Chartres Blue
(Stained glass)





Value discretized → Digital signal

Signal unit : 0 xor 1 (bit)

Boolean algebra : F xor T

Voltage level : L xor H

Multiple bit → binary operation → parallel signal

7.2 Logic gates

Digital signal=logic value $\rightarrow$ Logic operation : logic gates

De Morgan's laws: $\overline{x+y}=\bar{x}\cdot\bar{y},\ \overline{x\cdot y}=\bar{x}+\bar{y}$

		input					output			
t		t_1	t_2	$\cdots$	t_m		t_1	t_2	$\cdots$	t_m
Ch.	1	0	1	$\cdots$	f_{1m}	1	1	1	$\cdots$	q_{1m}
	2	1	0	$\cdots$	f_{2m}	2	0	1	$\cdots$	q_{2m}
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	n	0	1	$\cdots$	f_{nm}	l	0	1	$\cdots$	f_{lm}

Combinational logic $\rightarrow$ Truth table

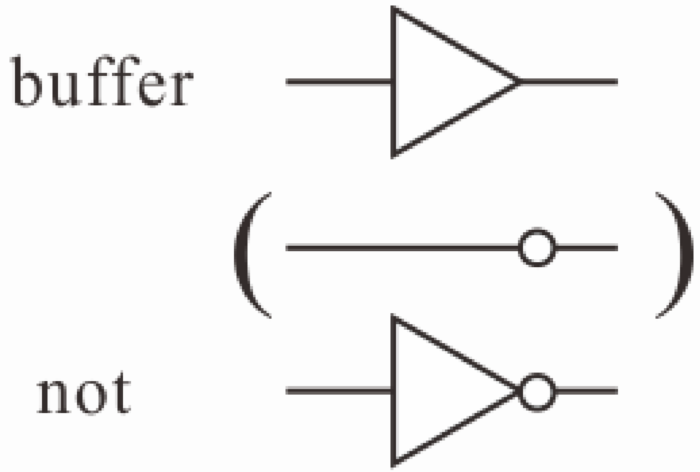
Sequential logic $\rightarrow$ Timing chart

7.2.1 Combinational logic: Single input gates

Truth table

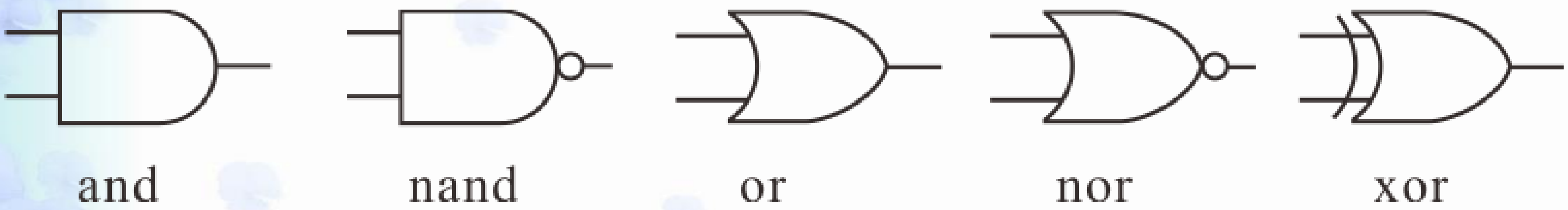
input	buffer	not
0	0	1
1	1	0

Circuit symbol



7.2.2 Combinational logic: Double input gates

input1	input 2	and	or	xor	nand
0	0	0	0	0	1
1	0	0	1	1	1
0	1	0	1	1	1
1	1	1	1	0	0



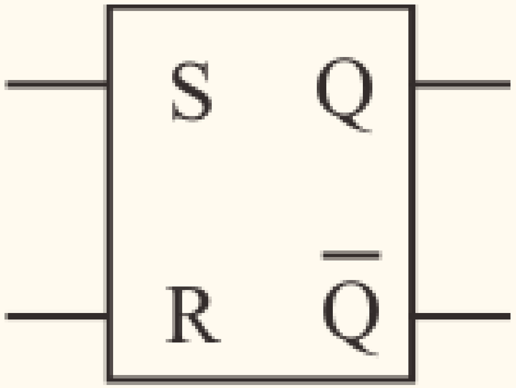
7.2.3 Sequential logic: Flip-Flop (FF)

RS (reset-set) Flip-Flop (FF)

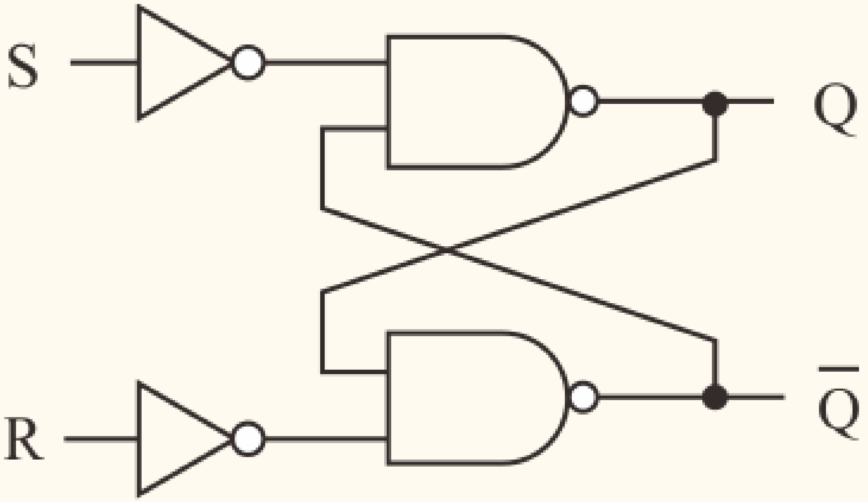
Truth table

S	R	Q	$\bar{Q}$	Response
0	0	Q	$\bar{Q}$	no change
0	1	0	1	reset
1	0	1	0	set
1	1	undetermined		

Symbol



Equivalent circuit with discrete gates

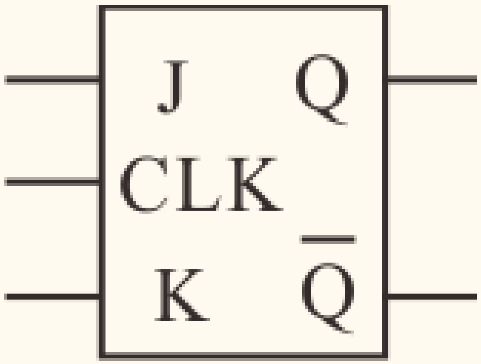


JK Flip-Flop

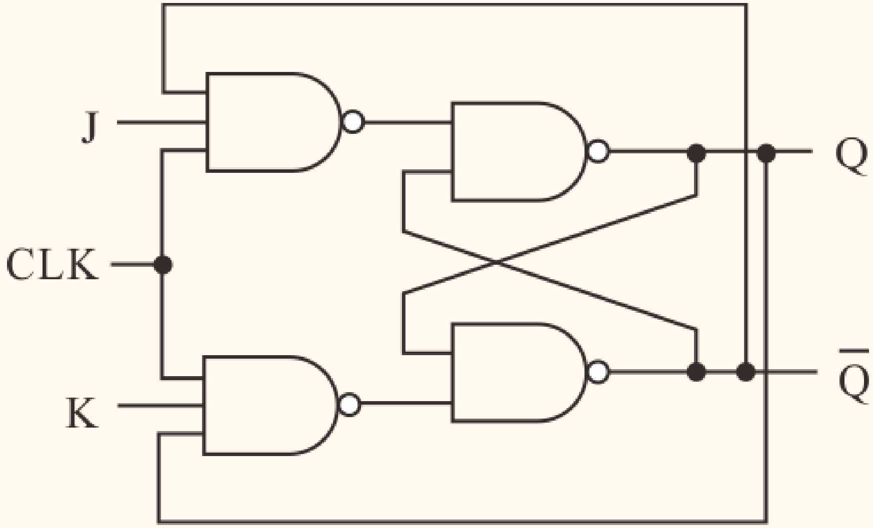
Truth table

J	K	Q	Q for the next CLK
0	0	0	0
0	0	1	1
0	1	–	0
1	0	–	1
1	1	0	1
1	1	1	0

Symbol



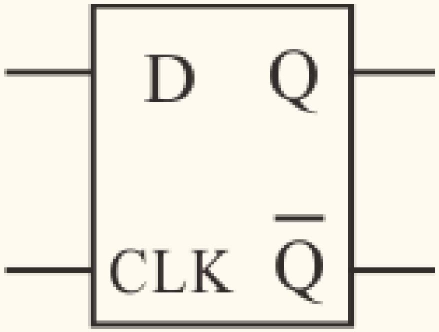
Equivalent circuit with discrete gates



7.2.3 Sequential logic: D-FF, T-FF

D-FF

Symbol

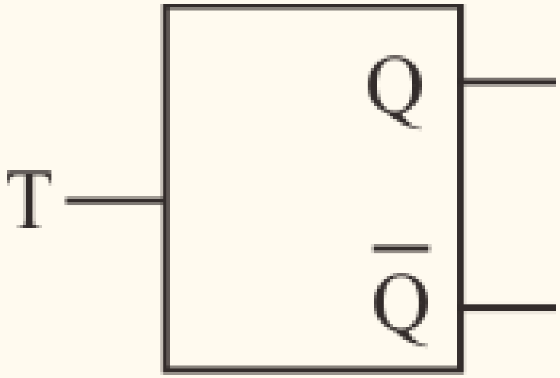


Truth table

D	CLK	Q
0	$\uparrow$	0
1	$\uparrow$	1
–	$\downarrow$	Q (hold)

T-FF

Symbol

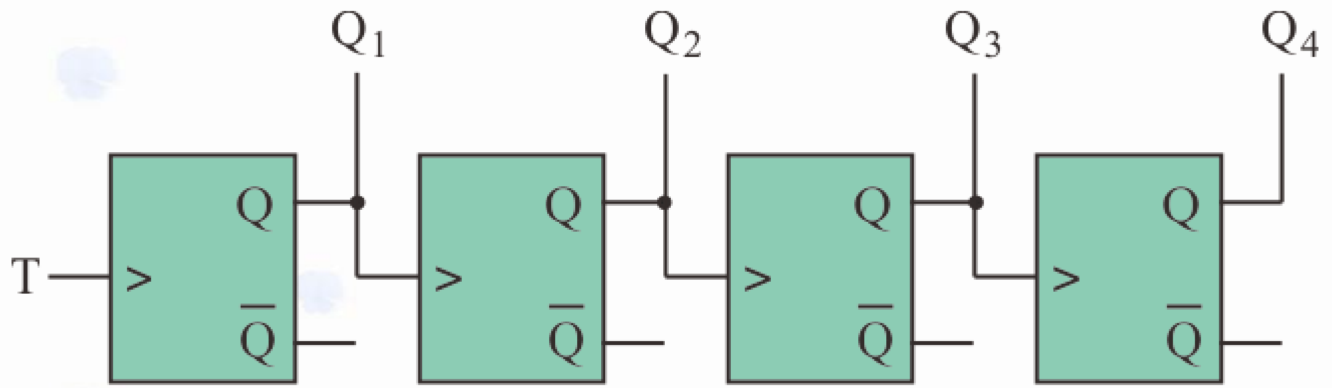


Truth table

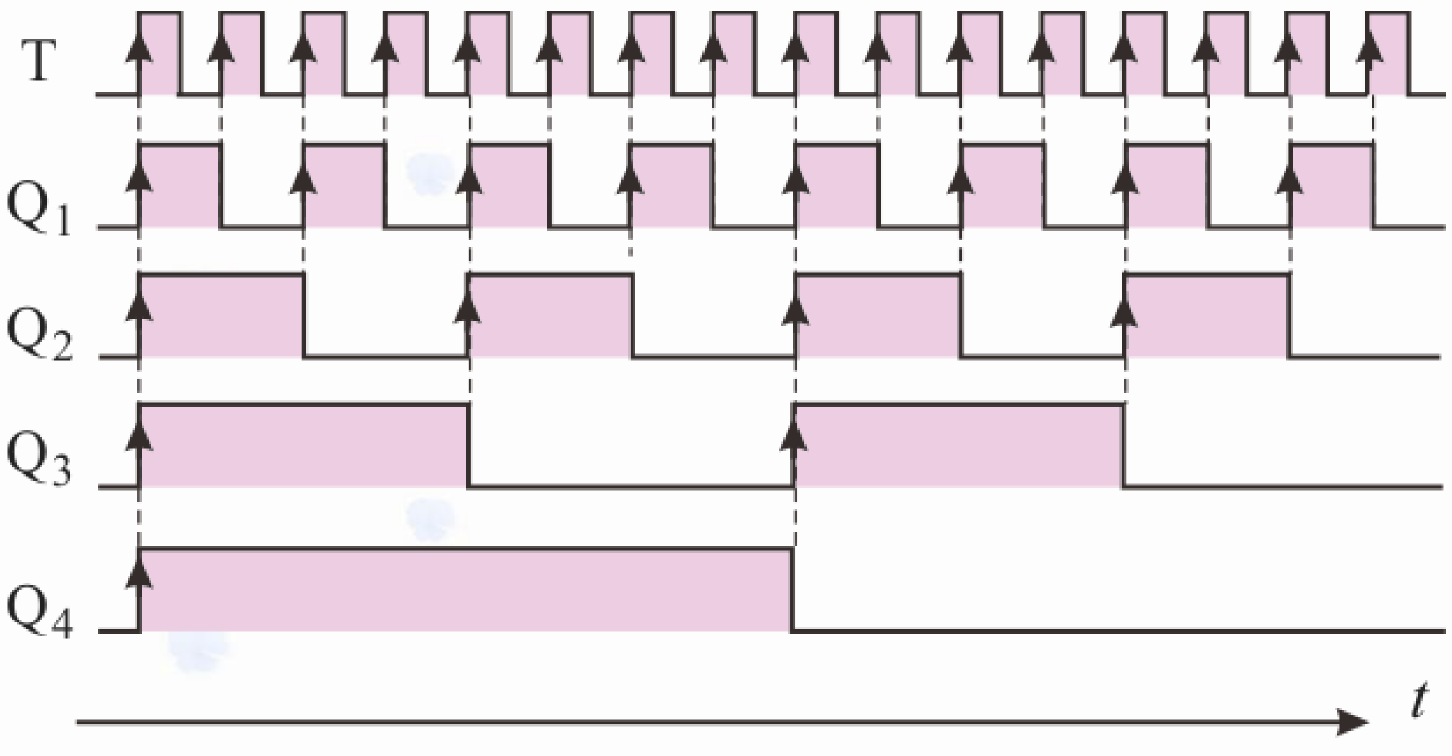
T	Q	Q
$\downarrow$	0	0
$\downarrow$	1	1
$\uparrow$	0	1
$\uparrow$	1	0

7.2.4 Sequential logic: Counters

Unsynchronized counter
(ripple counter)



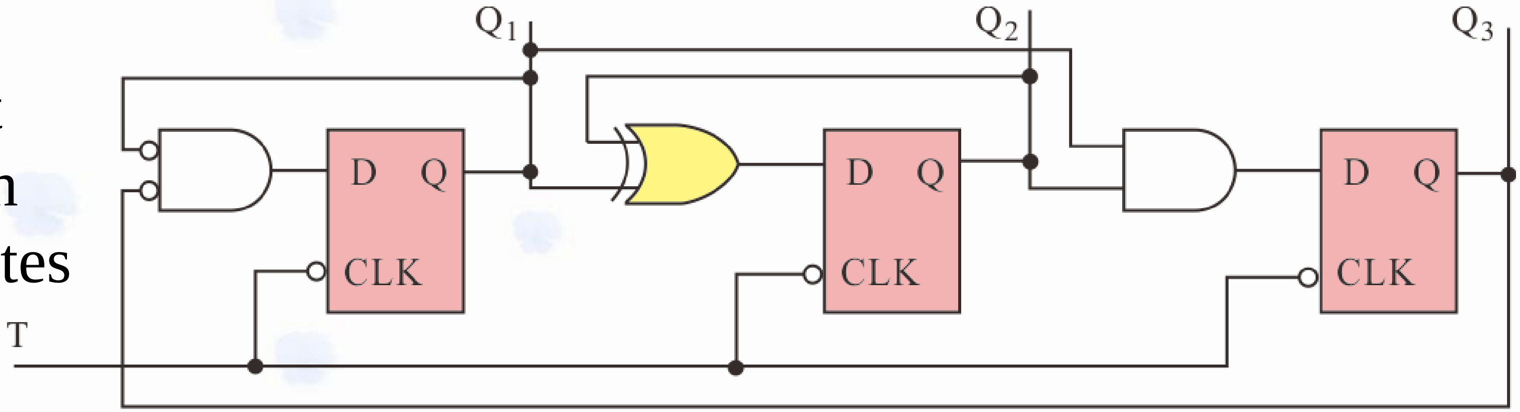
Timing chart



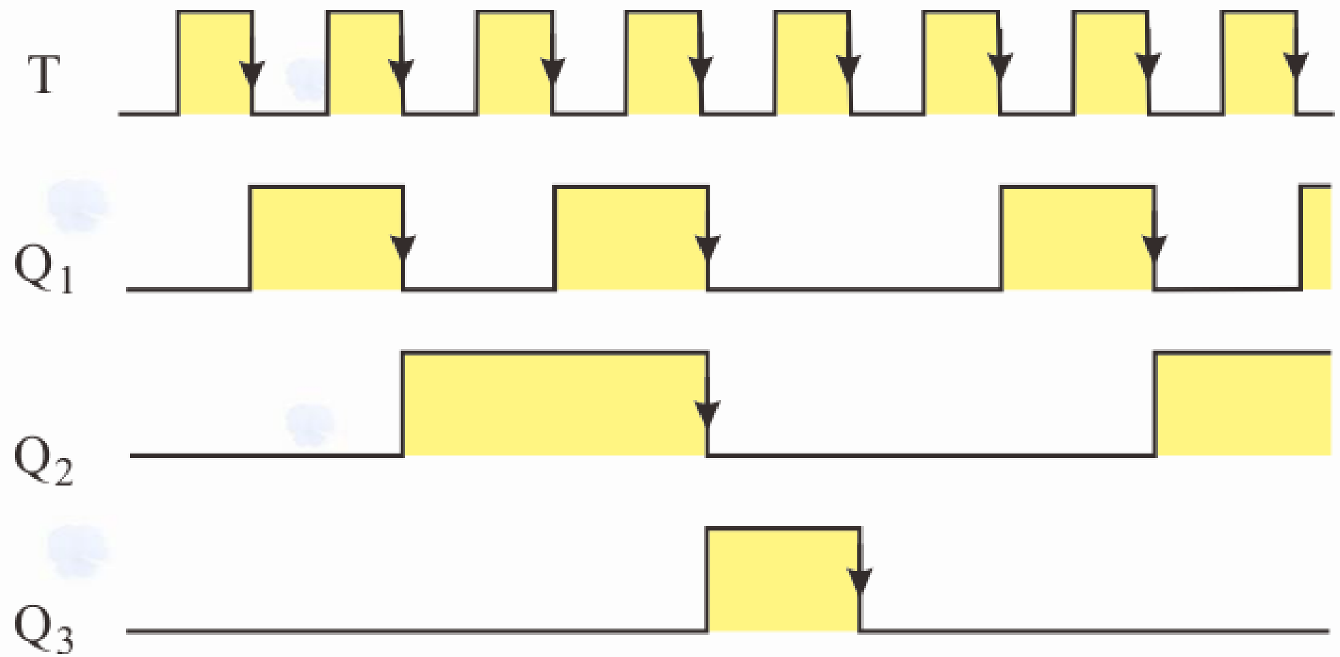
7.2.4 Sequential logic: Counters

Synchronized counter

Equivalent circuit with discrete gates

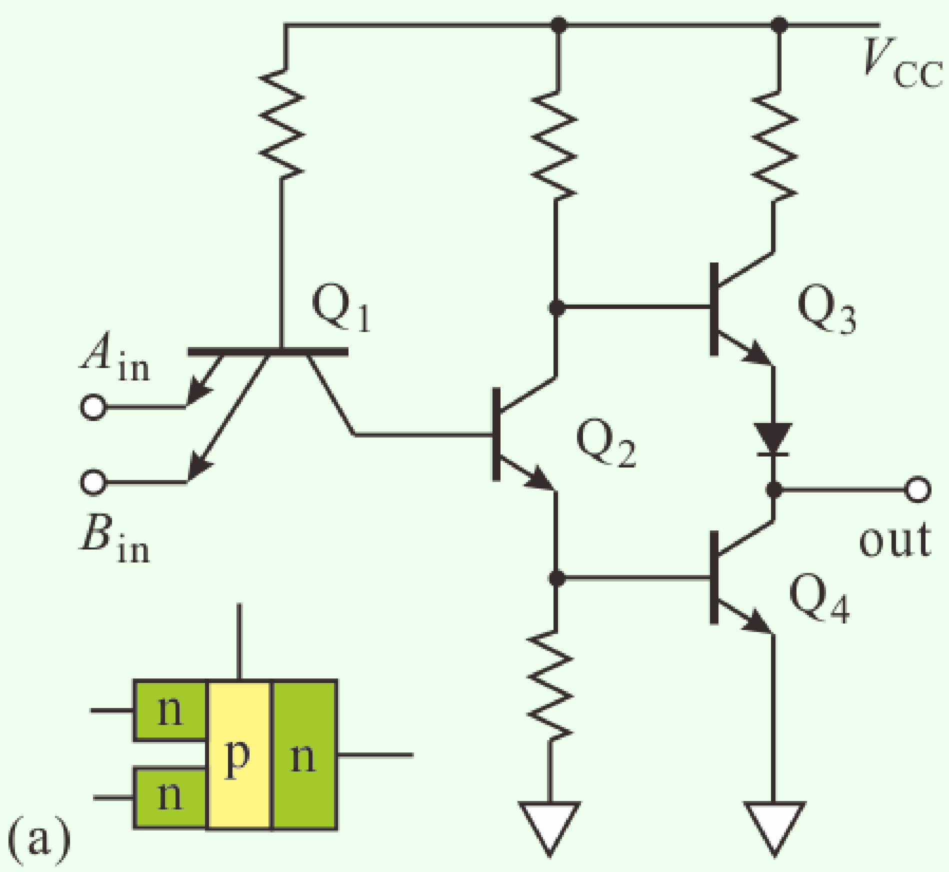


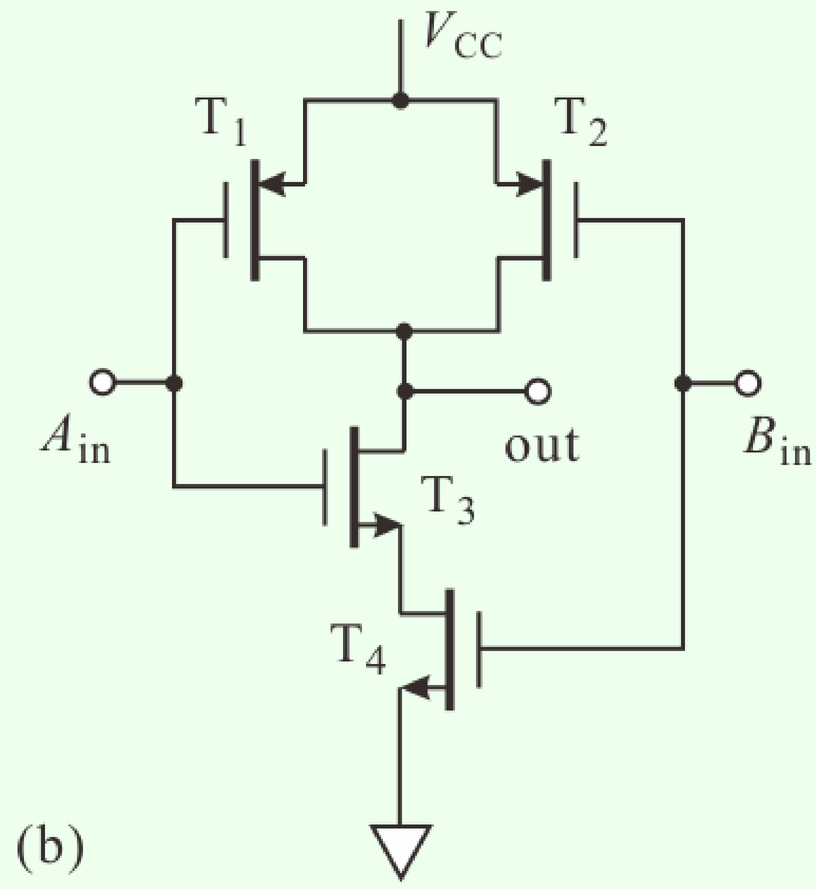
Timing chart



7.3 Implementation of logic gates

NAND gates



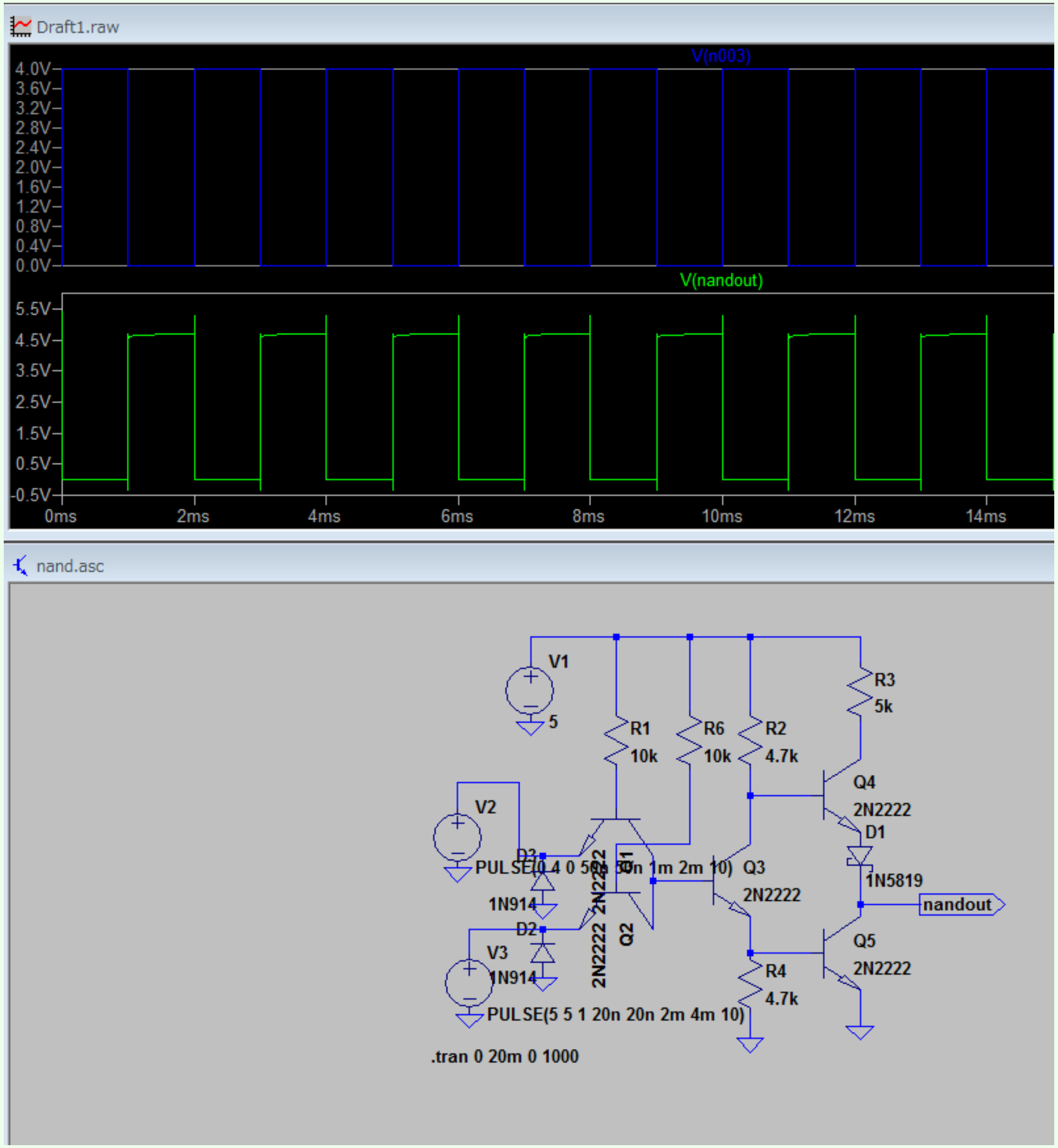


TTL (transistor-transistor logic)

CMOS (complimentary MOS)

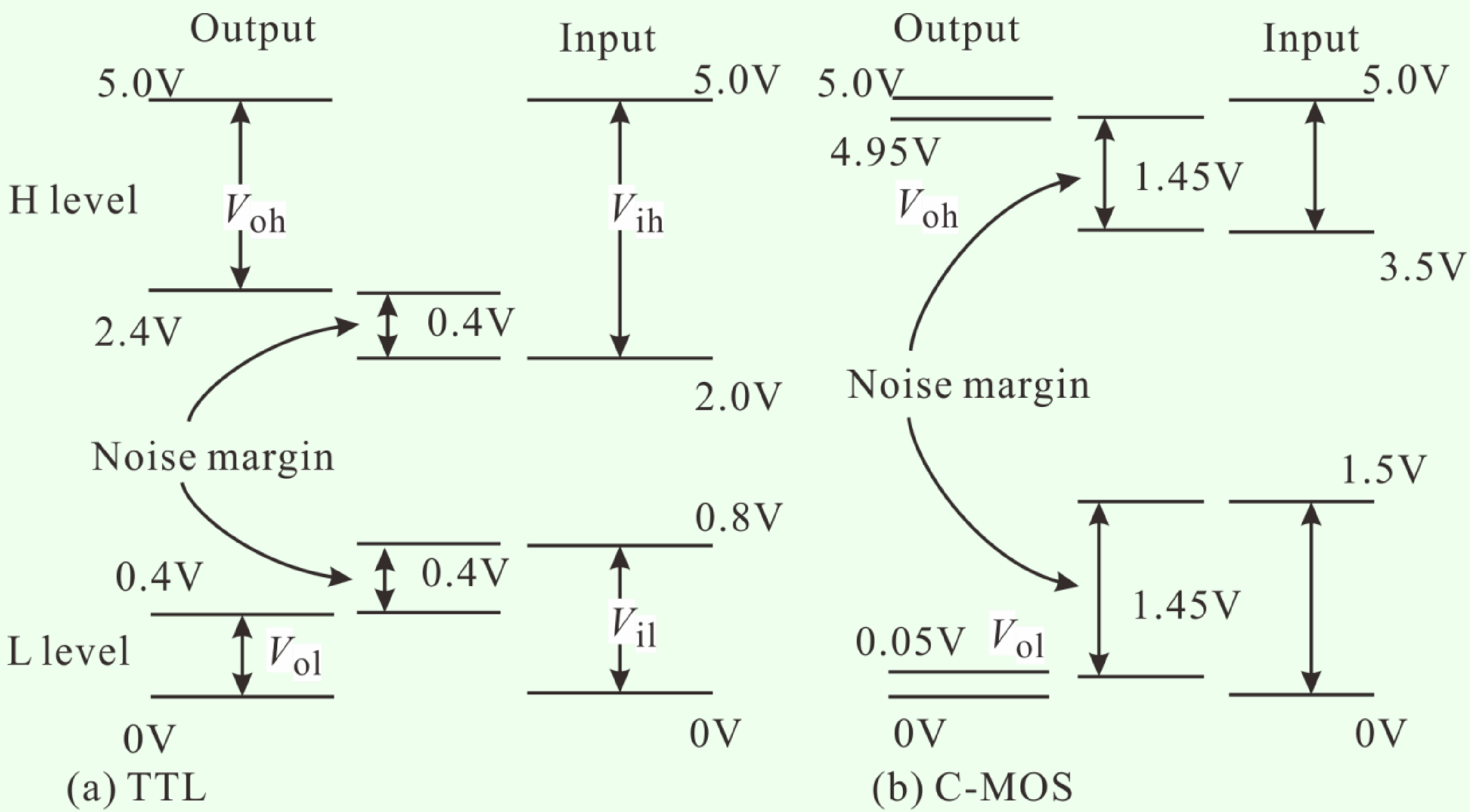
7.3 Implementation of logic gates

LT Spice simulation

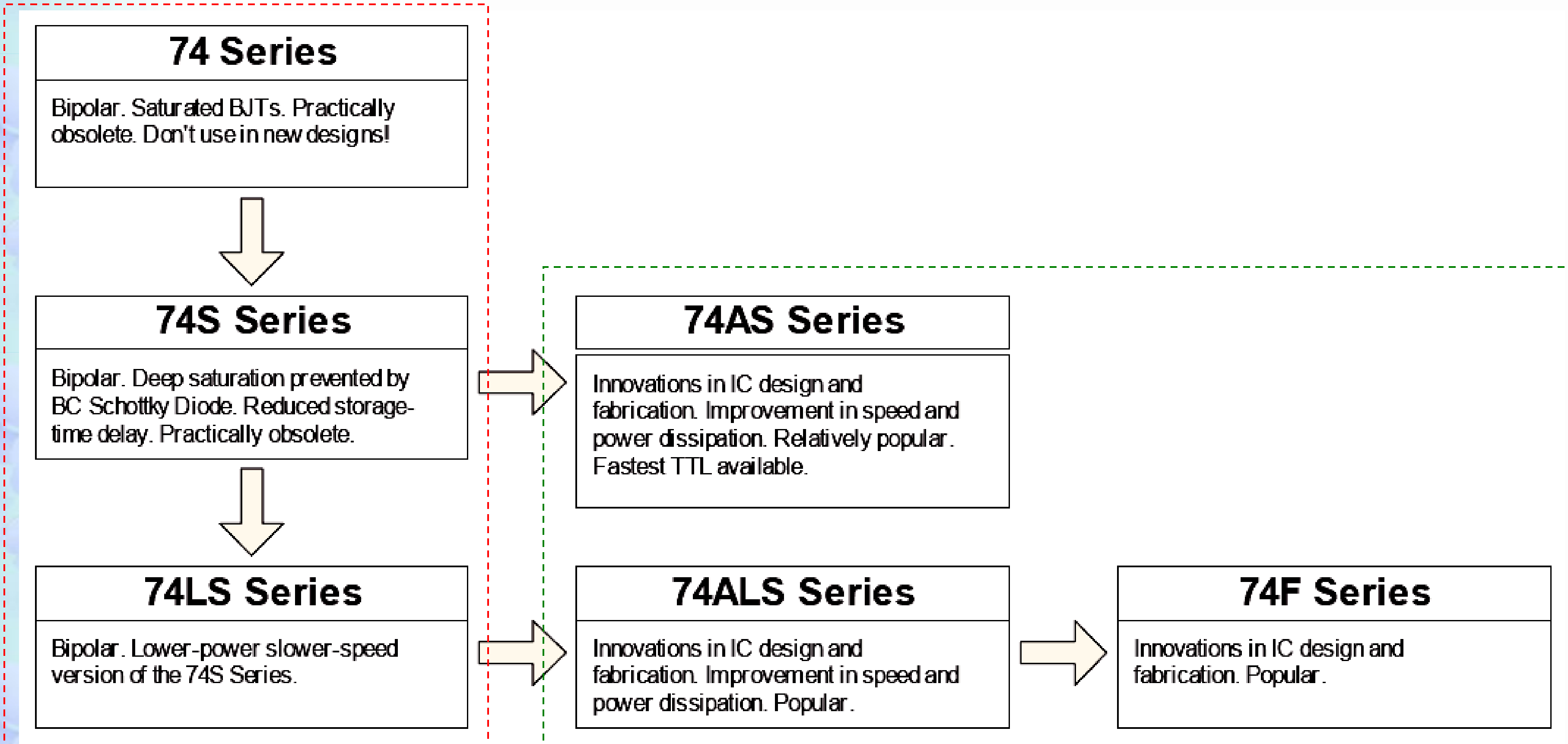


7.3 Implementation of logic gates

Voltage levels diagram



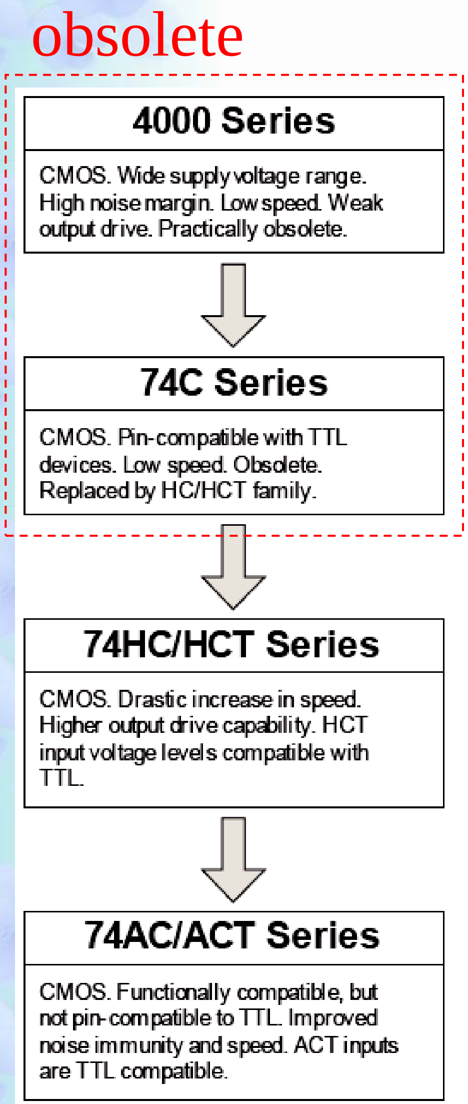
TTL logic family evolution



Legacy: don't use in new designs

Widely used today

CMOS logic family evolution



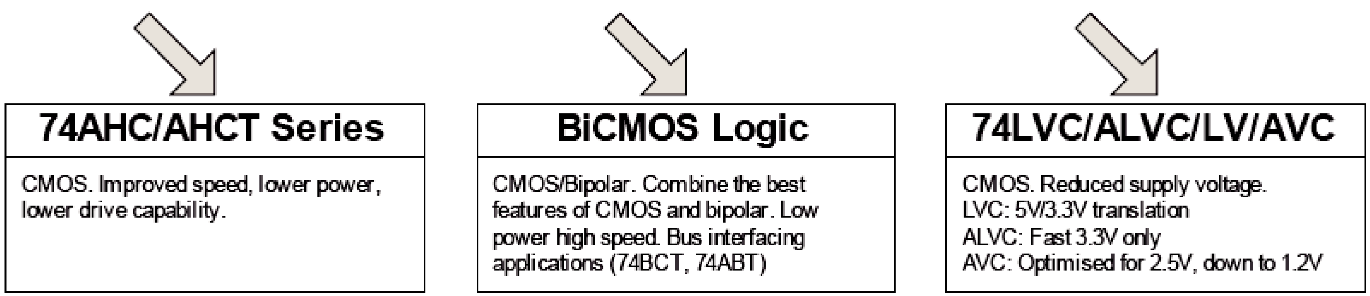
General trend:

• Reduction of dynamic losses through successively decreasing supply voltages:

12V → 5V → 3.3V → 2.5V → 1.8V

CD4000 LVC/ALVC/AVC

• Power reduction is one of the keys to progressive growth of integration



Summary

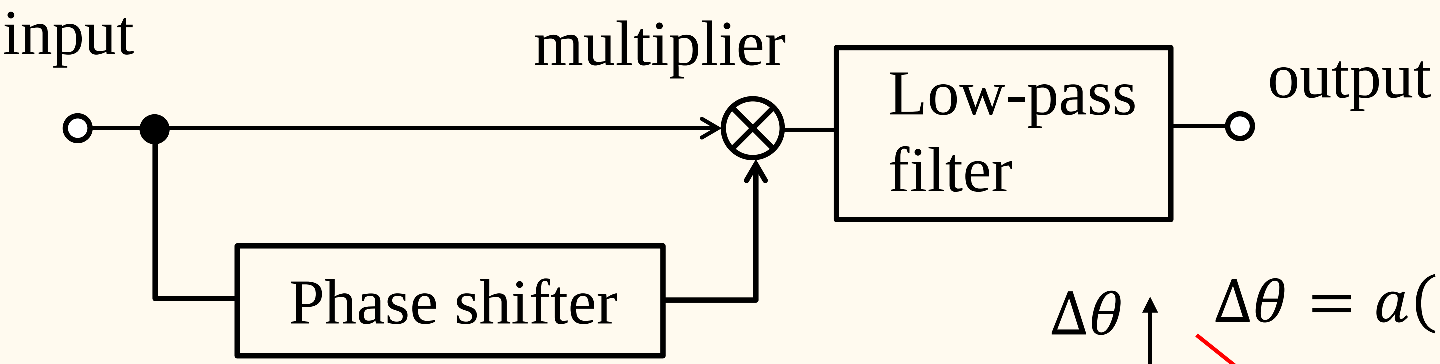
TTL

Logic Family	T_{PD}	$T_{rise/fall}$	$V_{IH,min}$	$V_{IL,max}$	$V_{OH,min}$	$V_{OL,max}$	Noise Margin
74	22ns		2.0V	0.8V	2.4V	0.4V	0.4V
74LS	15ns		2.0V	0.8V	2.7V	0.5V	0.3V
74F	5ns	2.3ns	2.0V	0.8V	2.7V	0.5V	0.3V
74AS	4.5ns	1.5ns	2.0V	0.8V	2.7V	0.5V	0.3V
74ALS	11ns	2.3ns	2.0V	0.8V	2.5V	0.5V	0.3V
ECL	1.45ns	0.35ns	-1.165V	-1.475V	-1.025V	-1.610V	0.135V
4000	250ns	90ns	3.5V	1.5V	4.95V	0.05V	1.45V
74C	90ns		3.5V	1.5V	4.5V	0.5V	1V
74HC	18ns	3.6ns	3.5V	1.0V	4.9V	0.1V	0.9V
74HCT	23ns	3.9ns	2.0V	0.8V	4.9V	0.1V	0.7V
74AC	9ns	1.5ns	3.5V	1.5V	4.9V	0.1V	1.4V
74ACT	9ns	1.5ns	2.0V	0.8V	4.9V	0.1V	0.7V
74AHC	3.7ns		3.85V	1.65V	4.4V	0.44V	0.55V

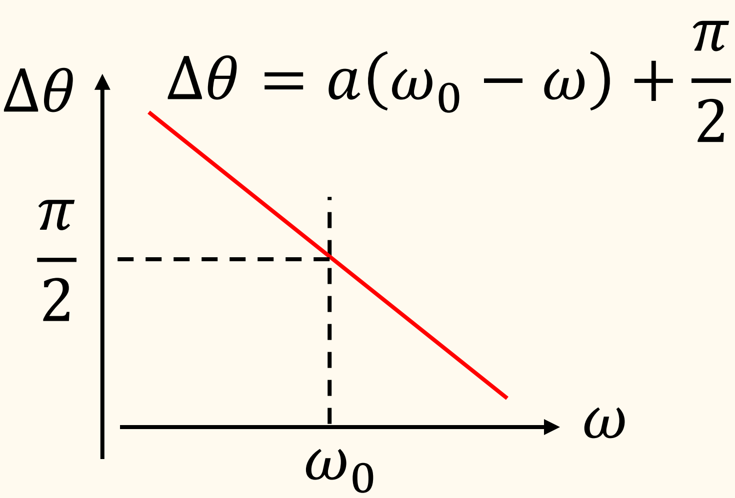
CMOS

Exercise F-1

Show that the following circuit works as a demodulator of frequency modulation (FM) signal (quadrature demodulator).



Here the phase shifter gives the shift proportional to the frequency difference between input and the career frequency ω_0. The shift at ω_0 is $\pi/2$ as shown in the right (this can be achieved with resonant circuits). The low-pass filter cuts components with frequencies as high as ω_0.



Exercise F-1

(hint) Assume the original signal $f(t)$ is much slower than the carrier $A\cos(\omega_0 t)$. Then the input can be approximated as

$$s(t) = A\cos\{[\omega_0 + k_f f(t)]t\}.$$

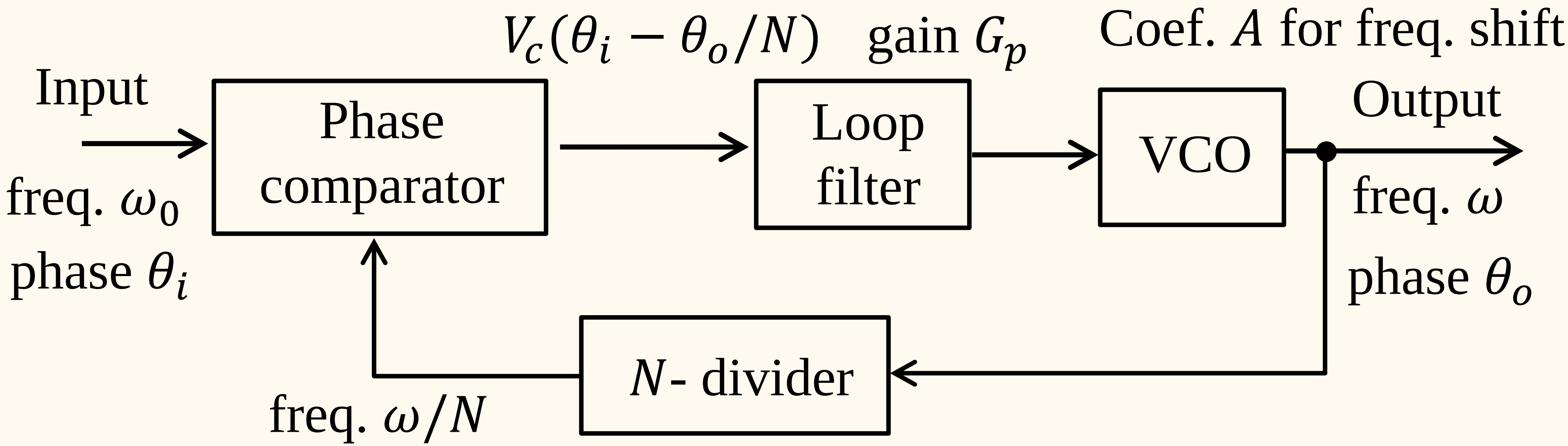
Then the phase shifter output is

$$q_{\mathrm{ps}}(t) = A\sin\{[\omega_0 + k_f f(t)]t - ak_f f(t)\}.$$

Taking product and high-frequency filtering gives …
(use $ak_f f(t) \ll 1$).

Exercise F-2

In the following phase lock loop (PLL) circuit, the initial ($t = 0$) oscillation frequency of voltage-controlled oscillator (VCO) ω deviates from $N\omega_0$ by $\Delta\omega$. Obtain the relaxation time of ω to $N\omega_0$.



(hint) Here we can put $\theta_i = 0$ hence input $= V_i \sin \omega_0 t$ without loosing generality. Similarly output $= V_o \sin[N\omega_0 t + \theta_o(t)]$. Now $\omega = N\omega_0 + d\theta_o/dt$ and it is easy to write $d\theta_o/dt$ with A, G_p, V_c, $\theta_o(t)$ and a constant.

Exercise F-3

Solve the difference equation below with z-transform.

$$\begin{cases} x(n) - 2x(n-1) = n & (n \geq 0) \\ x(n) = 0 & (n < 0) \end{cases}$$

(hint) z-transform of n is $\dfrac{z}{(z-1)^2}$ as in the table (slide no.14).

Then z-transform of $x(n)$: $X(z)$ is easily obtained. Inverse z-transform gives $x(n)$.

Answer sheet submission deadline: 11th Jan. 2017.