An example of answers for the final report of “Electronics"

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Here is an example of answers. There are many other possibilities.

1 DA conversion circuits

1.1 Resistance-ladder type

There is a difference at the right end from the one introduced in the lecture and the logic used there cannot be applied directly.

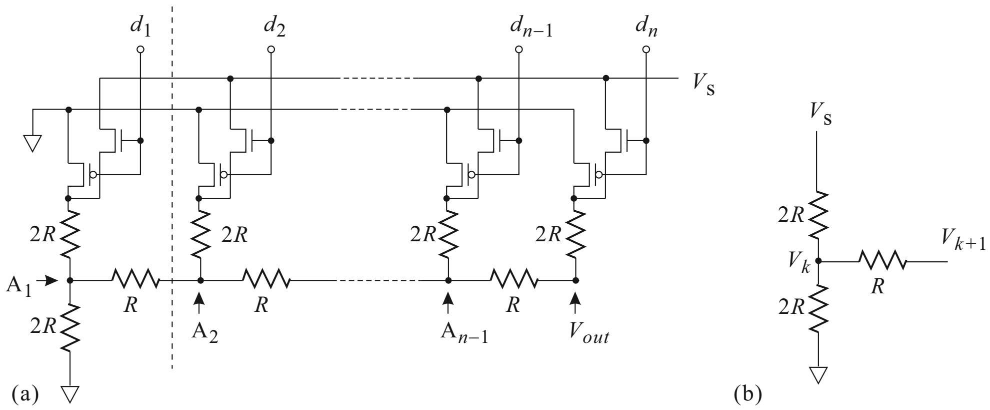


Figure 1: (a) A resistance-ladder type DA conversion circuit, in which the output should be amplified with high input impedance circuit. (b) An equivalent circuit around point A_k under the condition $d_k = 1,\ d_{l\neq k} = 0$.

There still is a common point, however, for the “resistance viewed from the right to the left at a joint. For example in Fig.1(a), when point d_1 is grounded, the MOS switch also grounds the resistance $2R$ standing from point A_1, and the total resistance from A_2 to the left is $2R$. This holds for point A_i as long as $d_k = 0$ $(k < i)$.

Next we write the voltage at point A_i $(1 < i < n)$ as V_i. Application of the Kirchhoff's first law to point A_i gives

$$(V_i - V_{i+1})/R + (V_i - V_{i-1})R + (V_i - V_s d_i)/2R = 0. \quad (1)$$

In case $d_i = 0$, $2V_{i+1} - 5V_i + 2V_{i-1} = 0$ holds. The characteristic equation $2x^2 - 5x + 2 = (2x - 1)(x - 2) = 0$ has solutions $x = 1/2, 2$ and the recurrence equation can be decomposed into $2V_{i+1} - V_i = a_0 2^i,\ V_{i+1} - 2V_i = b_0 2^{-i}$ (a_0 b_0: constants). Then the general solution has the form

$$V_i = D_1 2^i + D_2 2^{-i}, \quad (2)$$

where $D_1 = a_0/3$, $D_2 = -2b_0/3$ are constants.

Now we consider the condition $d_k = 1,\ d_{l\neq k} = 0$, under which we can determine D_1 and D_2 (note that they depend on n and k) from the boundary condition. First the situation at the right end gives $V_{\text{out}} = V_n = (2/3)V_{n-1}$. Substituting (2) into the above equation, we get the relation

$$D_2 = D_1 2^{2n+1}. \tag{3}$$

Next we consider the condition at $i = k$ (Fig.1(b)). At point A_k, the Kirchhoff's first law gives

$$(V_s - V_k)/(2R) - V_k/(2R) - (V_k - V_{k+1})/R = 0,$$
$$\therefore D_2 = \frac{V_s}{3} 2^k. \tag{4}$$

Equations (3) and (4) result in $D_1 = (V_s/3)2^{k-2n-1}$. Substituting the above results into eq.(2), we get the expression for the output voltage V_{out} as

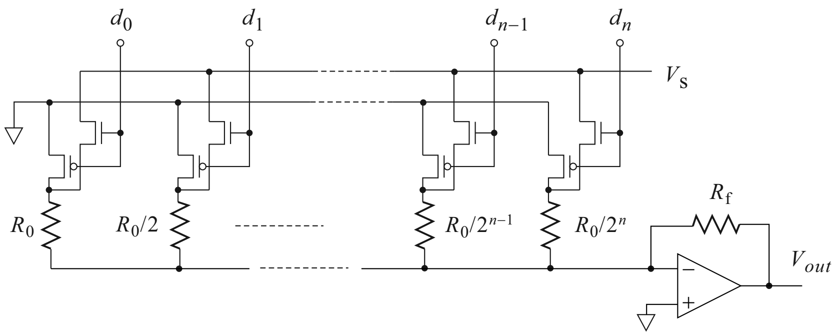
$$V_{\text{out}} = V_n = \frac{V_s}{3} 2^{k-2n-1} 2^n + \frac{V_s}{3} 2^k 2^{-n} = \frac{V_s}{2} 2^{k-n}. \tag{5}$$

Finally from the superposition theorem, for general sequence $\{d_k\}$ the following DA conversion output is obtained.

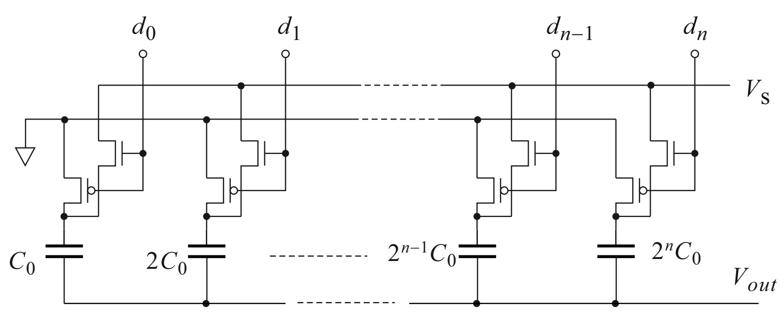
$$V_{\text{out}} = \frac{V_s}{2^{n+1}} \sum_{k=1}^{n} d_k 2^k. \tag{6}$$

1.2 Binary sequential resistance DA conversion circuit

If resistors with resistances $R_0/2^k$ $(k = 0, \cdots, n)$ are available, the currents through them when a fixed voltage V_s is applied, are $(V/R_0)2^k$. A current adder with an OP amp can easily accomplish the required circuit.



1.3 Binary sequential capacitance DA conversion circuit



In the Japanese version of the problem I miswrote "resistances" $\{2^k C_0\}$ $(k = 0, \cdots, n)$, but they are actually capacitances as in the English version. I hope the students reading this did not have any confusion. Also, I gave a comment "another C_0 capacitance" to some students' answers but this is just for stabilisation of charge in the output electrode and also for sample and hold. Just for answering the question this is not required. This does not affect the final scoring.

To answer the question, just like 1.2, we apply a voltage V_s to the capacitor with capacitance $2^k C_0$ and measure the total charge. In the above figure, the charge in the k-the capacitor is $(V_{\rm out} - d_k V_s)2^k C_0$ the total charge in the output electrode should be zero because there is no in/out of charge from it. Then,

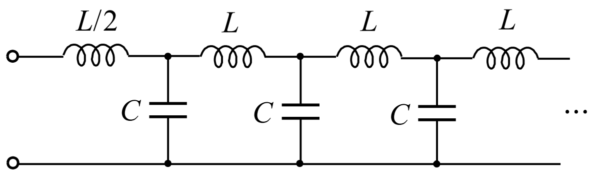
$$\sum_{k=0}^{n} (V_{\rm out} - d_k V_s)2^k C_0 = 0. \tag{7}$$

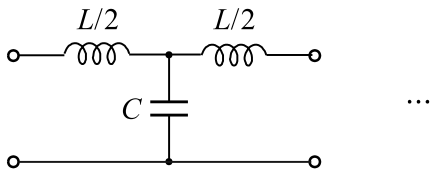
And we obtain

$$V_{\rm out} = \frac{V_s}{2^{n+1} - 1} \sum_{k=0}^{n} d_k 2^k. \tag{8}$$

Practically it is very hard to prepare stable capacitors with accurate capacitance over several decades and I do not know any real example of such DA converter. In serious practical application, we need to prepare many additional circuits, *e.g.*, sample and hold capacitors, discharging switches etc.

2 Distributed constant circuit





The title "distributed constant circuit" is not appropriate. Here the continuum limit is not taken and "circuit with repetition" or "periodic circuit" are appropriate expressions. There are various ways for taking unit of repetition. Anyway one of the general properties of periodic circuits tells us that the boundary between transparent and opaque (decaying) frequency regions is the same as that of the unit of repetition.

Let us take the unit of repetition as a T-shaped circuit composed of two inductors with inductance $L/2$ and a capacitor with capacitance C as shown in the left. The F-matrix of a unit is given as

$$F(\omega) = \begin{pmatrix} i\omega L \left(1 - \frac{\omega^2 LC}{4}\right) & 1 - \frac{\omega^2 LC}{2} \\ 1 - \frac{\omega^2 LC}{2} & i\omega C \end{pmatrix}. \tag{9}$$

The F-matrix for cascade connection of n units is, then, written as $F_n(\omega) = F^n(\omega)$.

Let $q_\pm(\omega)$ be the two eigenvalues of $F(\omega)$, then F can be written by unitary matrix U and diagonal matrix G, which has $q_\pm(\omega)$ as the diagonal elements, as $F = U^{-1}GU$. Then F^n is expressed as

$$F_n(\omega) = U^{-1}GUU^{-1}GU \cdots U^{-1}GU = U^{-1}G^n U = U^{-1} \begin{pmatrix} q_+^n(\omega) & 0 \\ 0 & q_-^n(\omega) \end{pmatrix} U. \tag{10}$$

Specific expression of $q_\pm(\omega)$ is

$$q_\pm(\omega) = 1 - \frac{\omega^2 LC}{2} \pm \sqrt{\omega^2 LC \left(\frac{\omega^2 LC}{4} - 1\right)}. \tag{11}$$

When the inside the square root is negative, that is,

$$\omega < \frac{2}{\sqrt{LC}}, \tag{12}$$

then

$$|q_\pm| = \sqrt{\left(1 - \frac{\omega^2 LC}{2}\right)^2 - \omega^2 LC \left(\frac{\omega^2 LC}{4} - 1\right)} = 1. \tag{13}$$

Namely in this frequency region, in the limit $n \to \infty$ in (10), the voltage (current) that propagate through the circuit does not decay but has phase rotation. Hence eq.(12) represents the condition of transparent region and $\omega > 2/\sqrt{LC}$ corresponds to attenuation region. $2/\sqrt{LC}$ is the cut-off frequency.

The above can be reproduced in the calculation under the concept of image impedance. The image impedance (introduced in lecture no.4 slide no.12) for the present case is

$$Z_1 = Z_2^{-1} = \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2 LC}{4}\right)}. \tag{14}$$

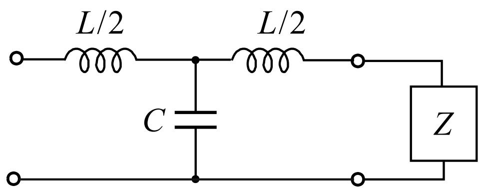
From the expression given in slide no.14 in lecture no.4 and the properties of hyperbolic functions, the F-matrix for n-cascade circuit is given as

$$F(\omega)^n = \begin{pmatrix} \cosh n\theta & Z_1 \sinh n\theta \\ Z_1^{-1} \sinh n\theta & \cosh n\theta \end{pmatrix}. \tag{15}$$

This means, again, the transmission range of the total circuit is the same as that of the unit circuit. And the condition is equivalent to the image attenuation constant (introduced in slide no.13) is zero. That is

$$1 = \left|\frac{V_1 J_1}{V_2 J_2}\right| \Leftrightarrow 1 = \sqrt{\left|\frac{V_1 J_1}{V_2 J_2}\right|} = \left|\sqrt{AD} + \sqrt{BC}\right| = \left|1 - \frac{\omega^2 LC}{2} + \sqrt{\left(\frac{\omega^2 LC}{4} - 1\right)\omega^2 LC}\right|,$$

which is the same as from (11) to (13).



From the F-matrix in (15) the two-wire impedance for the left end terminals is nothing but the image impedance (14). Or, the same can be obtained in the following way. To get the result for $n \to \infty$, we consider the circuit being resemble to the Dyson equation, that is, the impedance Z should be the same as the cascade connection of a single unit and Z itself. Namely

$$Z = i\omega\frac{L}{2} + \frac{1}{i\omega C + \frac{1}{i\omega L/2 + Z}}. \tag{16}$$

Then Z is given as

$$Z = \sqrt{\frac{L}{C}\left(1 - \omega^2 \frac{LC}{4}\right)}, \tag{17}$$

which is, as expected, the same as the image impedance. I wrote some wrong comments that the edge inductance should give some imaginary part, but that is the case for the edge inductance of L and in the present problem, it is $L/2$ and there is no imaginary part. I apologise this but anyway this did not cause any deduction for the right answer. As can be seen in the above, the impedance is zero at the cut-off frequency, *i.e.* this is the resonance point.

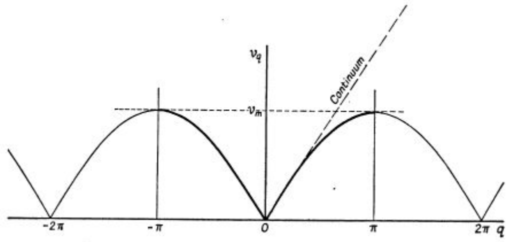
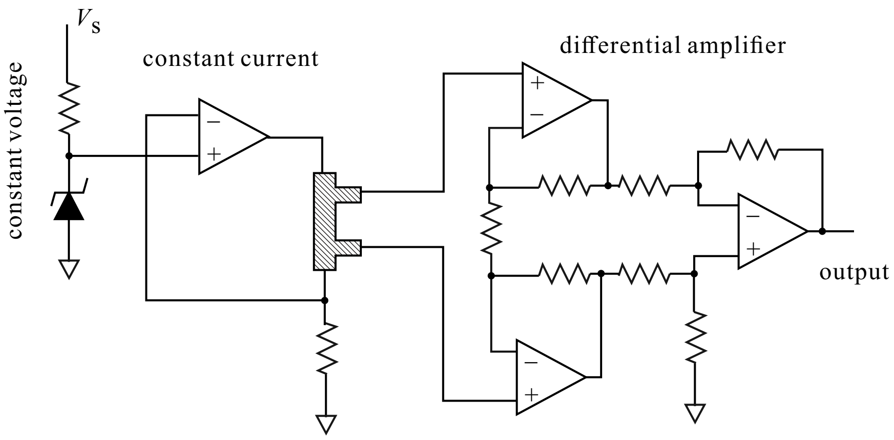


FIG. 2. Frequency versus wave number for a linear chain.

Such a discrete transmission line with repetition unit composed of L and C converges into a distributed constant transmission line in the limit of infinitesimal unit, which line does not have such a cut-off frequency. This is, of course, because the limit is taken under the condition that $L/C = Z_0^2$ is kept constant. Under that the cut-off frequency $2/\sqrt{LC}$, on the other hand, diverges. There are many similar phenomena, in which some cut-off frequencies appear with discretising continuous spaces. For example, the above figure is the dispersion relation of acoustic phonon in one-dimensional lattice, taken from J. M. Ziman "Electrons and Phonons". Around zero-frequency ($\nu = 0$), the acoustic phonon has no dispersion (massless) because the wave does not "see" the lattice discreteness. With increasing the wavenumber q (or the frequency), the linear relation changes into a convex curve forming cut-off frequency $\nu_{\rm th}$. The cut-off wavelength is just a half of the lattice period and there is a resonance between the phonon and the lattice. This also corresponds to the Nyquist frequency in the sampling theorem. In the continuum limit, as indicated in the figure as "Continuum", the cut-off frequency or the resonant point escapes to infinite.

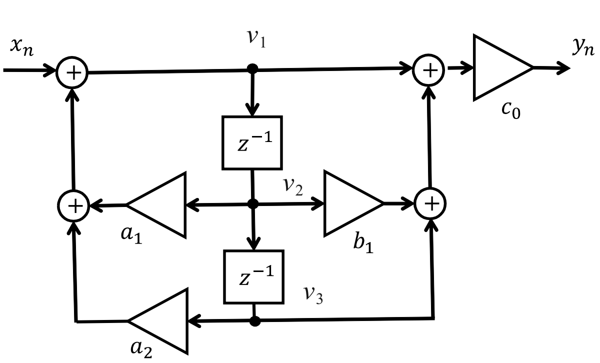
3 OPamp circuit



The above is an example of circuits which satisfy the requirements. The Zener diode creates a constant voltage, which is converted into a constant current. In this case the electrostatic potential of the sample is apart from the ground and the voltage across the sample is taken by a differential amplifier with high input impedance (in the lecture this is introduced as "instrumentation amplifier"). In realistic low-temperature measurement, the thermopower imbalance should be cancelled out by changing the direction of driving current or by using low frequency lock-in technique.

4 Digital filter

4.1 Input-output relation



This is a famous circuit, which works as a notch IIR filter. The solution can be obtained straightforwardly by following the block diagram. The vertices are symbolised $v_{1,2,3}$ as shown in the left. Then the relations

$$
\begin{aligned}
v_2 &= v_1 z^{-1}, \\
v_1 &= x_n + a_1 v_2 + a_2 v_3, \\
v_3 &= v_2 z^{-1}, \\
y_n &= c_0(v_1 + b_1 v_2 + v_3)
\end{aligned} \tag{18}
$$

are obtained. The upper three equations give

$$(1 - a_1 z^{-1} - a_2 z^{-2}) v_1 = x_n. \tag{19}$$

Then from (18), and $y_n z^{-k} = y_{n-k}$,

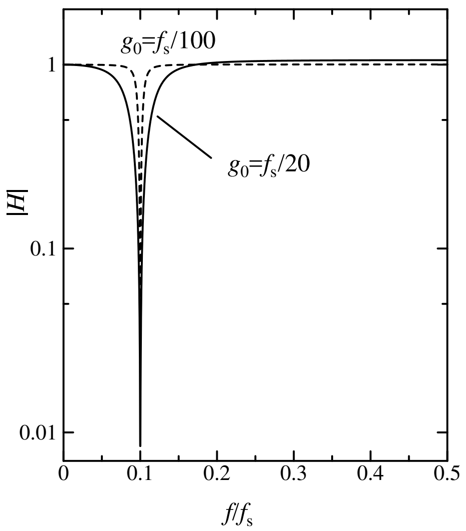
$$\begin{aligned} & y_n - a_1 y_{n-1} - a_2 y_{n-2} \\ & \quad = c_0[(1 - a_1 z^{-1} - a_2 z^{-2}) v_1 + b_1(1 - a_1 z^{-1} - a_2 z^{-2}) v_1 z^{-1} + (1 - a_1 z^{-1} - a_2 z^{-2}) v_1 z^{-2}] \\ & \quad = c_0(x_n + b_1 x_{n-1} + x_{n-2}). \end{aligned} \tag{20}$$

This is the relation required in the problem.

4.2 Frequency response

From (20),

$$y_n - a_1 y_{n-1} - a_2 y_{n-2} = (1 - a_1 z^{-1} - a_2 z{-2}) y_n = c_0(x_n + b_1 x_{n-1} + x_{n-2}) = c_0(1 + b_1 z^{-1} + z^{-2}) x_n,$$



then the transfer function $H(z)$ is written as

$$H(z) = \frac{c_0(1 + b_1 z^{-1} + z^{-2})}{1 - a_1 z^{-1} - a_2 z{-2}}. \tag{21}$$

The frequency dependence is obtained by substituting $z = e^{i\omega\tau}$ into the above.

$$|H(e^{i\omega\tau})| = \left| \frac{c_0(1 + b_1 e^{-i\omega\tau} + e^{-2i\omega\tau})}{1 - a_1 e^{-i\omega\tau} - a_2 e^{-2i\omega\tau}} \right|. \tag{22}$$

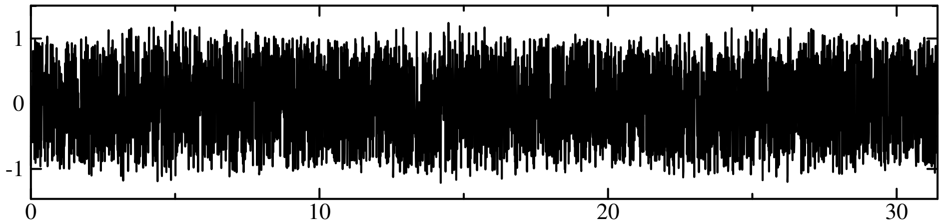
Before the analysis of (22), let us write a simple program on a convenient language like Scilab, and draw $|H(e^{i\omega\tau})|$ as a function of ω. As shown in the left, it (transmission coefficient or gain, if you prefer to call) has a sharp dip at $f/f_\mathrm{s} = 0.1$. Such a filtering characteristic is called **notch filter** or **band stopping filter**. When g_0 is set to $f_\mathrm{s}/100$, as indicated by the broken line, the dip sits at the same position but is narrowed. From this numerical test, we infer that g_0 determines the dip width whereas f_0 is the dip position.

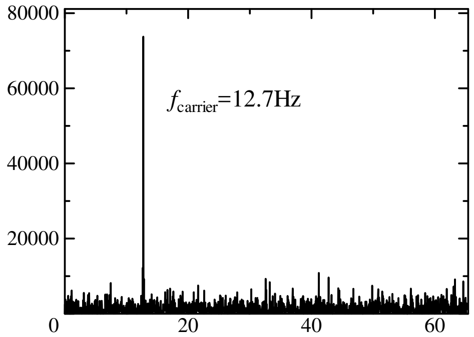
Now let us check eqs.(21), (22) along the inference. From the relation $c_0 = (1 - a_1 - a_2)/(2 + b_1)$, $H = 1$ for $f = 0$. Namely c_0 is the normalization constant for the condition. Next the numerator can be written as $1 - (e^{if_0} + e^{-if_0})e^{-if} + e^{-2if} = (e^{-if} - e^{-if_0})(e^{-if} + e^{-if_0})$, where we adopt simplified notation f_0 as $2\pi f_0 \tau$ etc. This actually has the only zero at $f = f_0$.

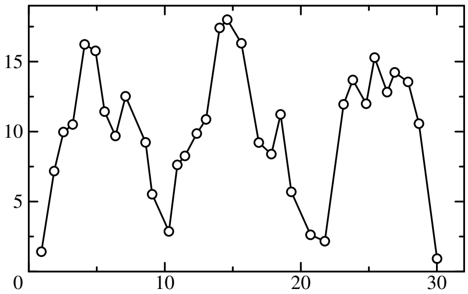
In the same notation, the denominator is $(e^{g_0/2 - if} - e^{-if_0})(e^{g_0/2 - if} + e^{-if_0})$, which does not have zero (pole of H) due to the factor $e^{g_0/2}$. To see the behavior in the vicinity of f_0, we write $\Delta f \equiv f - f_0$ and check the anomaly around f_0 in the factor $e^{g_0/2 - if} - e^{-if_0}$. We expand the term with $\Delta f \sim 0$ and $g_0 \ll \pi$ and take to the first order of Δf, g_0 to obtain

$$|e^{g_0/2 - if} - e^{-if_0}| \sim |g_0/2 - i\Delta f| = \sqrt{(\Delta f)^2 + \left(\frac{g_0}{2}\right)^2}.$$

This represents a dip at $\Delta f = 0$ with width g_0. The above results justify what we inferred from the numerical check.







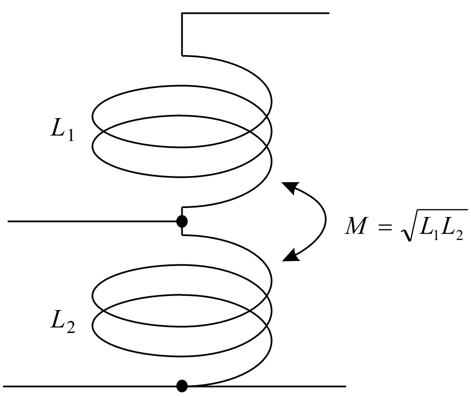
5 Discrete Fourier transform

This problem can be straghtfowardly solved just with a little perspiration on numerical calculation (if you possess a convenient data analysis program, this can easily be accomplished).

The above is the given signal, which is, at the first sight, like white noise.

But in the FFT power spectrum, as shown in the upper left figure, a clear peak appears at 12.7 Hz if we assume the unit of x-parameter in the data is second. Hence the signal carrier is there. Now we repeat the FFT on a narrow window and check the amplitude at 12.7 Hz with shifting the center of the window. Then a triangular wave pattern over three periods should be reproduced with fluctuation due to the difference in analysis parameters.

6 Impedance matching



I thought the problem is easy for most of the students to solve but in practice many were suffered from this to my surprise. The main reason seems to be that I forgot to explain how we treat in the simplest approximation the situation of “take an intermediate tap to a coil". I apologize for lacking the explanation though I thought is a kind of common sense. I introduced the technique of matching transformer in the lecture for impedance matching. At high frequencies, the winding numbers of coils are small and we often adopt such a middle tap method instead of transformer. Many students who tried this problem considered equivalent circuits with deviding the coil into two at the middle tap. At this point the fact that these two coils have magnetic fluxes in common slipped out. Such “independent coils" approximation makes the calculation even more complicated and leads to wrong answers.

As in the above figure, when two coils have magnetic flux in common (interlinkage flux), we need to consider the mutual inductance M. In the simplest approximation that there is no leakage in flux (close coupling), $M = \sqrt{L_1 L_2}$, where L_1, L_2 are the inductances for the two independent coils. Here the total inductance L is

$$L = L_1 + L_2 + 2M = L_1 + L_2 + 2\sqrt{L_1 L_2} = (\sqrt{L_1} + \sqrt{L_2})^2. \tag{23}$$

This is consistent with the simplest approximation that the “inductance is proportional to the square of winding number".

Then the voltage (electromotive force) ratio of the tap side to the total coil side is that of inductance L_2/L. Hence the condition is the same for a matching transformer and

$$\frac{L_2}{L} = \frac{N_1^2}{N_2^2} = \frac{50}{800} = \frac{1}{16} \quad \therefore \frac{N_1}{N_2} = \frac{1}{4} = 0.250.$$

Most of the students reached the right answer for the values of L and C. First the ratio of resonance frequency to the width (Q-value) should be

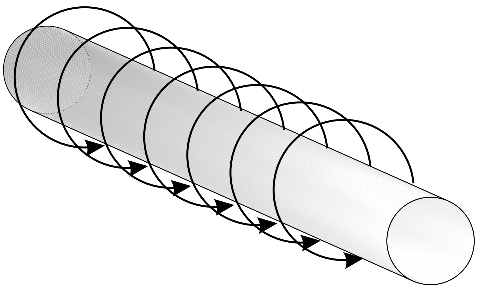
$$Q = \frac{\omega_0}{|\omega_1 - \omega_2|} = \frac{85}{10} \approx 2\pi \times 85 \times 10^6 \times 800 \times C,$$

giving

$$C = 1.99 \times 10^{-11}\ \mathrm{F} = 19.9\ \mathrm{pF}.$$

Next from the resonance condition

$$L = \frac{1}{C\omega_0^2} = \frac{1}{1.99 \times 10^{-11} \times (2\pi \times 85 \times 10^6)^2} = 1.76 \times 10^{-7}\ \mathrm{H} = 0.176\ \mu\mathrm{H}.$$



An interesting question here may be “in problem no.2, we linearly devide or connect the inductances. Why there is such difference from eq.(23)?" This is because when we consider LC unit periodic circuit model, we assume transverse electromagnetic field (TEM) mode for propagation. As in the left figure, in such case, the magnetic flux does not have longitudinal component, which corresponds to the mutual inductance between the coils in series. Hence we can add or devide the inductances in linear approximation. This is an example that we need to remember we are treating electromagnetc field even in the circuit approximation.