# Electric Circuits for Physicists
# 電子回路論　第10回

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# Erratum

Impedance matching condition for transmission line termination

Characteristic impedance $Z_0$, termination impedance $Z$: $Z = Z_0$

Assumption: Propagating mode → $Z_0$ : real

$Z_0$ is, in a sense, transverse impedance and does not cause energy dissipation.

Minimization of energy loss at the output impedance $Z_o$ : $Z = Z_o^*$

$$R_{\mathrm{in}} = S_{11} + \frac{S_{12}S_{21}R_{\mathrm{L}}}{1 - S_{22}R_{\mathrm{L}}} \qquad R_{\mathrm{out}} = S_{22} + \frac{S_{12}S_{21}R_{\mathrm{S}}}{1 - S_{11}R_{\mathrm{S}}}$$

Matching condition: $R_{\mathrm{L}} = R_{\mathrm{out}}^*, \quad R_{\mathrm{S}} = R_{\mathrm{in}}^*$

# Outline

# 6.1 Fluctuation

Quantity $x$, fluctuation $\delta x = x - \bar{x}$

$$\overline{(\delta x)^2} = \overline{(x-\bar{x})^2} = \overline{x^2} - \bar{x}^2 \quad (\overline{\delta x} = 0)$$

$g(x)$: distribution function of $x$

Fourier transform: $$u(q) = \mathscr{F}\{g(x)\} = \int_{-\infty}^{\infty} g(x) e^{ixq} \frac{dx}{\sqrt{2\pi}}$$

$u(t)$ : **characteristic function** of the distribution

From Taylor expansion, any moment can be obtained as

$$\overline{x^n} = \frac{\sqrt{2\pi}}{i^n}\left[\frac{d^n}{dq^n}u(q)\right]_{q=0}$$

Moments to high orders → reconstruction of $g(x)$

# Power Spectrum

Consider probability sets in the interval [0,$T$).

$$x_j(t) = \sum_{n=1}^{\infty} (a_{jn} \cos \omega_n t + b_{jn} \sin \omega_n t), \quad \omega_n = \frac{2n\pi}{T}$$

$$\mathscr{P}_{jn} = (a_{nj} \cos \omega_n t + b_{nj} \sin \omega_n t)^2$$

$$\langle \mathscr{P}_n \rangle = \frac{1}{2} \langle a_n^2 + b_n^2 \rangle$$ ∵ cross product terms are averaged out

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Random process:

Gaussian distribution $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma^2}\right]$$

$$\overline{(\delta x)^2} = \sigma^2, \quad \overline{\left(\sum_{j=1}^{m} \delta x_j\right)^2} = m\sigma^2$$

Then $\overline{\langle \mathscr{P}_n \rangle} = \sigma_n^2$ (non-Markovian)

# Power Spectrum

## Power spectrum $G(\omega)$

Frequency band width $\delta\omega$ : separation between two adjacent frequencies

$$\delta\omega = \omega_{n+1} - \omega_n = \frac{2(n+1)\pi}{T} - \frac{2n\pi}{T} = \frac{2\pi}{T}$$

$$G(\omega_n)\frac{\delta\omega}{2\pi} = \overline{\langle \mathscr{P}_n \rangle} \; (= \sigma_n^2)$$

$$\overline{\langle x^2(t) \rangle} = \sum_{n=1}^{\infty} \overline{\langle \mathscr{P}_n \rangle} \quad (\overline{x(t)} = 0)$$

$$= \sum_n G(\omega_n)\frac{\delta\omega}{2\pi} \to \int_0^\infty G(\omega)\frac{d\omega}{2\pi}$$

# 6.1.1 Fluctuation-Dissipation Theorem

$$\omega_0 \equiv 1/\sqrt{LC} \qquad Z(i\omega) = \frac{R(\omega_0^2 - \omega^2) + i\omega_0^2\omega L}{\omega_0^2 - \omega^2},$$

$$Y(i\omega) = \frac{\omega_0^2 - \omega^2}{R(\omega_0^2 - \omega^2) + i\omega_0^2\omega L}$$

$V(t)$ noise power spectrum $\rightarrow G_v(\omega)$

$$G_v(\omega) = 4k_{\mathrm{B}}T\mathrm{Re}[Z(i\omega)]$$

$$G_v(\omega) = 4k_{\mathrm{B}}TR$$

Johnson-Nyquist noise
Thermal noise

# 6.1.2 Wiener-Khintchine Theorem

Self-correlation function $C(\tau) = \overline{\langle x(t)x(t+\tau)\rangle}$

$$= \overline{\sum_{n,m} \langle [a_n \cos\omega_n t + b_n \sin\omega_n t][a_m \cos\omega_m (t+\tau) + b_m \sin\omega_m (t+\tau)]\rangle}$$

$$= \frac{1}{2}\sum_n \overline{\langle a_n^2 + b_n^2\rangle} \cos\omega_n\tau = \sum_n \overline{\langle \mathscr{P}_n\rangle} \cos\omega_n\tau$$

$$= \int_0^\infty G(\omega)\cos\omega\tau \frac{d\omega}{2\pi}$$

$$C(\tau) = \int_0^\infty G(\omega)\cos\omega\tau \frac{d\omega}{2\pi}$$

$$G(\omega) = 4\int_0^\infty C(\tau)\cos\omega\tau d\tau$$

Wiener-Khintchine theorem

# 6.1.2 Wiener-Khintchine Theorem

Example)

$$C(\tau) = \exp\left(-\frac{\tau}{\tau_0}\right)$$

$$G(f) = 4\int_0^{\infty} e^{-\tau/\tau_0}\cos(2\pi f\tau)d\tau = \frac{4\tau_0}{1+(2\pi f\tau_0)^2}$$

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## 6.1.4 Nyquist Theorem

Mode density on a transmission line with length *l*

$$\omega_n = c^* k_n = \frac{2\pi n c^*}{l} \qquad \therefore \quad \delta\omega = \frac{2\pi c^*}{l}$$

Bidirectional → Freedom × 2

Bose distribution $$f(\hbar\omega, T) = \frac{1}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1}$$

# 6.1.4 Nyquist Theorem

Thermal energy density per freedom

$$\frac{\hbar\omega}{\exp(\hbar\omega/k_{\mathrm{B}}T)-1} \sim \frac{\hbar\omega}{1+(\hbar\omega/k_{\mathrm{B}}T)-1} = k_{\mathrm{B}}T \quad (k_{\mathrm{B}}T \gg \hbar\omega)$$

Thermal energy density in band $\Delta\omega$

$$2\frac{\Delta\omega}{\delta\omega}k_{\mathrm{B}}T = \frac{2k_{\mathrm{B}}Tl}{2\pi c^*}\Delta\omega$$ , a half of which flows in one-direction

Energy flowing out from the end:

$$\frac{k_{\mathrm{B}}Tl}{2\pi c^*}\Delta\omega \times \frac{1}{l} \times c^* = k_{\mathrm{B}}T\Delta f \quad (2\pi f = \omega)$$

equals the energy supplied from the noise source.

$$\overline{J^2}R = k_{\mathrm{B}}T\Delta f, \quad \overline{V^2} = 4Rk_{\mathrm{B}}T\Delta f$$

$$\sqrt{\overline{J^2V^2}} = 2k_{\mathrm{B}}T\Delta f$$

→ Noise Temperature

## Single Electron

Time domain: $\delta$-function approximation

$$J_e(t) = e\delta(t - t_0)$$
$$= e\int_{-\infty}^{\infty} e^{2\pi i f(t-t_0)} df = 2e\int_0^{\infty} \cos\left[2\pi f(t - t_0)\right] df$$

Uniform 2$e$ in frequency domain: fluctuation at each frequency

Coherent only at $t = t_0$

Current fluctuation density for infinitesimal band $df$

$$\delta J = d\sqrt{\langle J_e^2 \rangle} = \frac{2e}{\sqrt{2}} df = \sqrt{2} e df$$

## Double Electron

$$\overline{\langle \delta j^2 \rangle} = (j_p + j_q e^{i\phi})(j_p + j_q e^{-i\phi}) = j_p^2 + j_q^2 + 2 j_p j_q \cos\phi$$

$\phi$: coherent phase shift → averaged out

$$\overline{\langle \delta j^2 \rangle} = j_p^2 + j_q^2 = 2 \times (\sqrt{2}e)^2 df$$

## $N$-Electron

$$\overline{\langle \delta J^2 \rangle} = N \times 2e^2 df = 2e\overline{J} df \quad (\overline{J} = eN)$$

Quantum mechanical correlation → Modification from random

# 6.1.5 Shot Noise

Example: pn junction

Current-Voltage characteristics: $J(V) = J_0 \left[ \exp\left( \frac{eV}{k_B T} \right) - 1 \right]$

Differential resistance $r_d = \left( \frac{dJ}{dV} \right)^{-1} = \left[ \frac{eJ_0}{k_B T} \exp\left( \frac{eV}{k_B T} \right) \right]^{-1} = \frac{k_B T}{e} \frac{1}{J + J_0}$

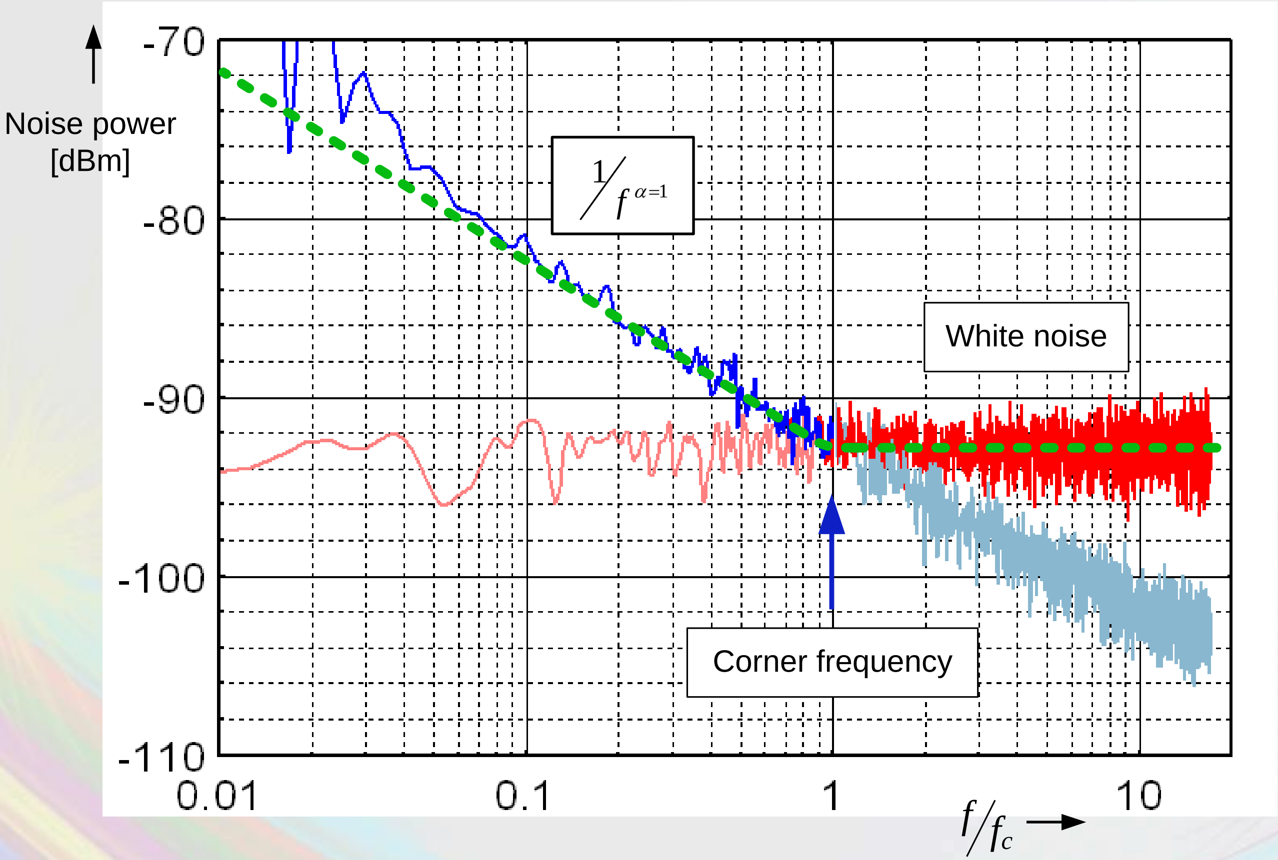
$$J \gg J_0 \;\rightarrow\; r_d \sim k_B T / eJ$$

$$\overline{\langle (\delta J)^2 \rangle} = 2e \frac{k_B T}{e r_d} df = 4 k_B T \frac{1}{2 r_d} df$$

$$(\delta V)^2 = 4 \frac{r_d}{2} k_B T \Delta f$$

# 6.1.6 1/f noise

$$(\delta V)^2 = K J^a R^2 \frac{\Delta f}{f^\alpha}$$



# "Unit" of Noise

Noise: Power spectrum per frequency

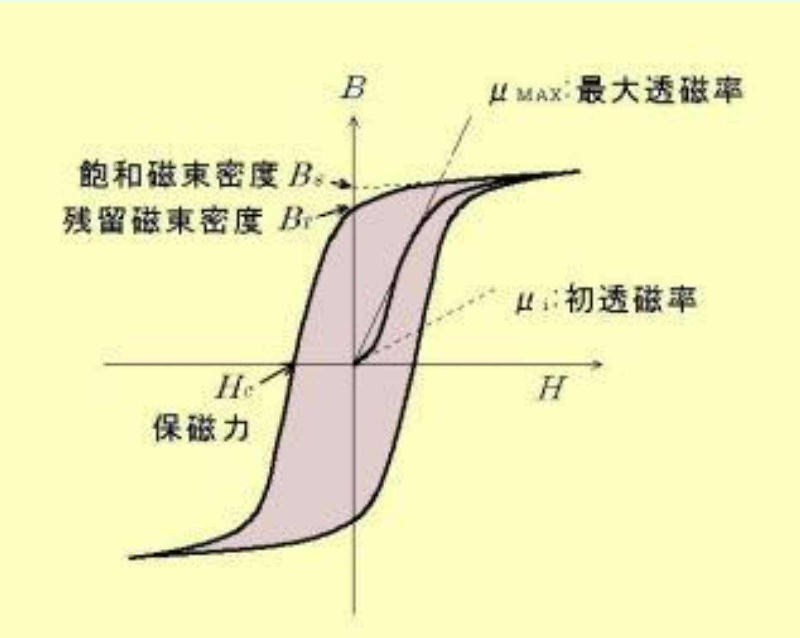
$$\overline{j_n^2} = \overline{\delta J^2}/\Delta f, \quad \overline{e_n^2} = \overline{\delta V^2}/\Delta f$$

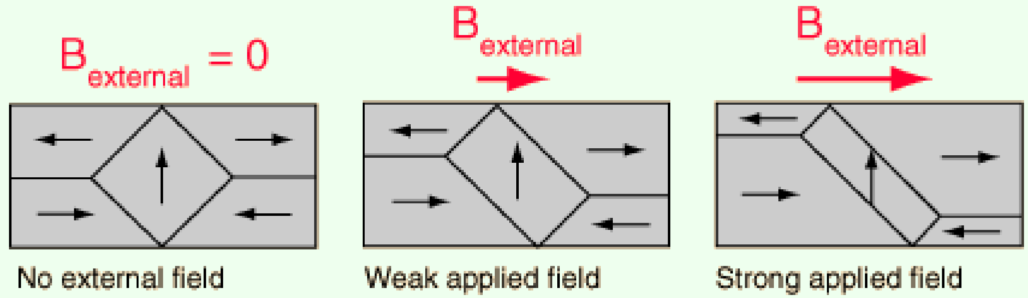
⬇

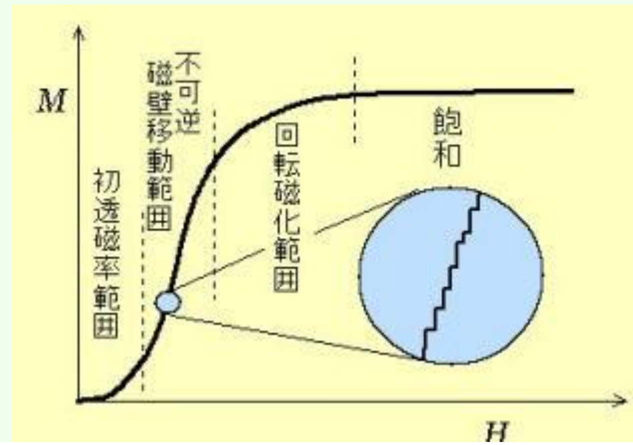
$$\text{unit of } \sqrt{\overline{j_n^2}}, \quad \sqrt{\overline{e_n^2}}$$

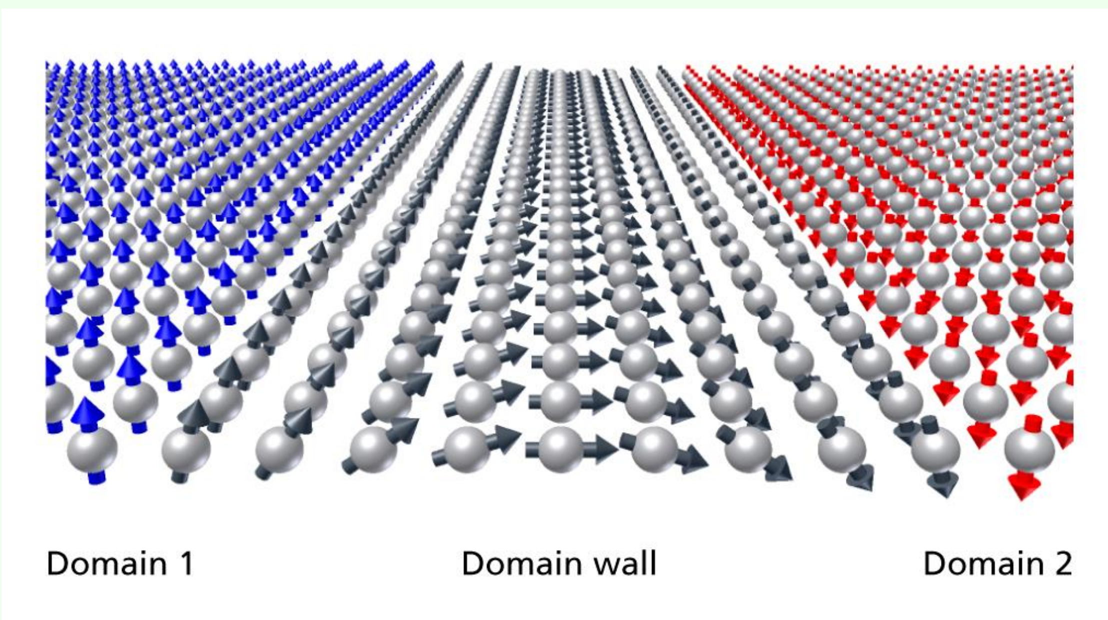
$$\mathrm{A}/\sqrt{\mathrm{Hz}}, \quad \mathrm{V}/\sqrt{\mathrm{Hz}}$$

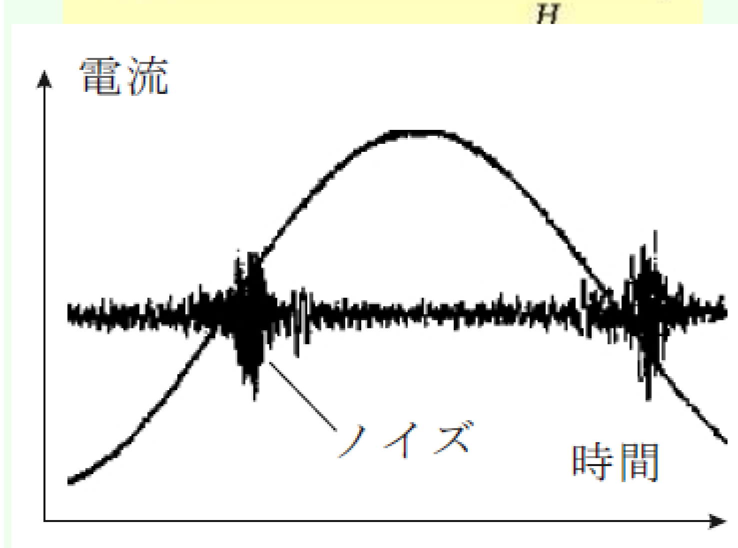
# Other noises: Barkhausen noise



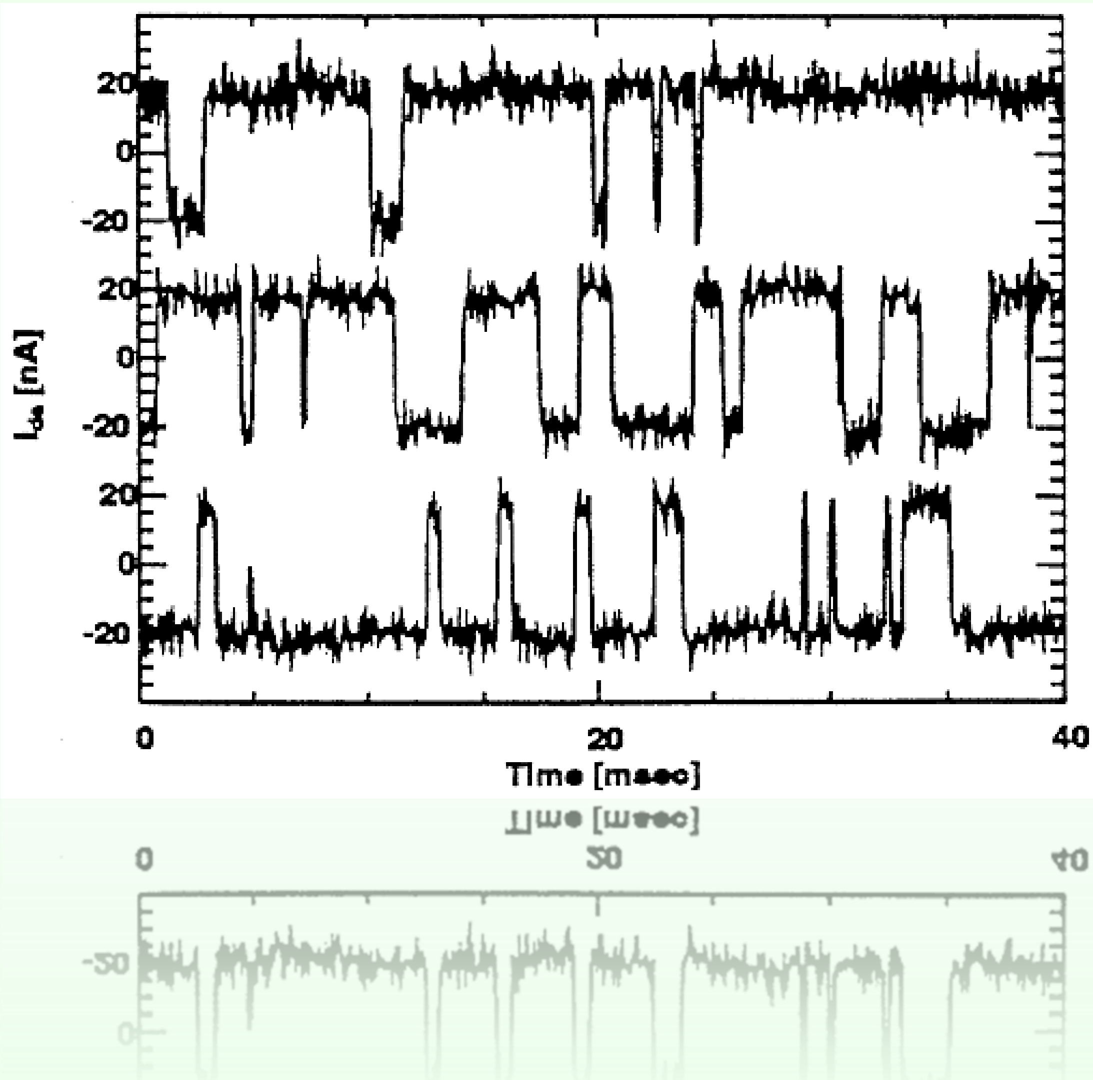



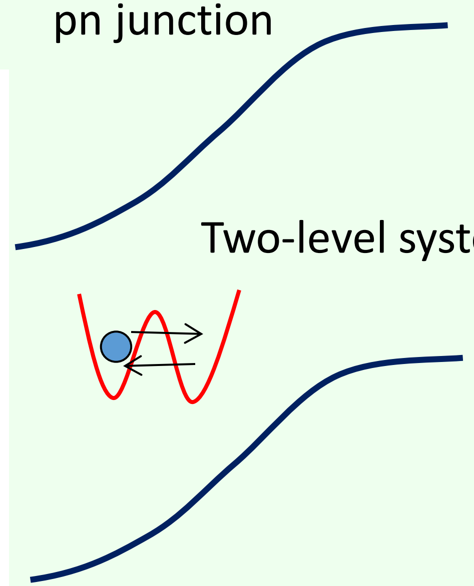
# Popcorn noise

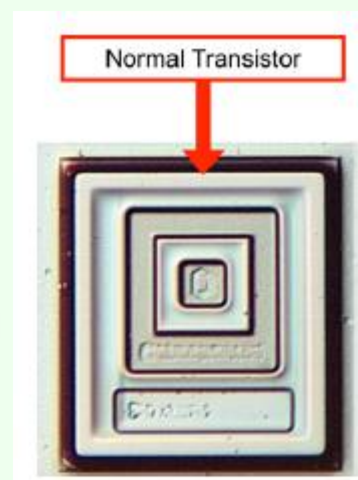
Popcorn noise, Burst noise

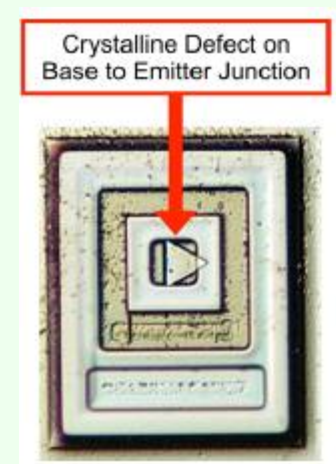



pn junction

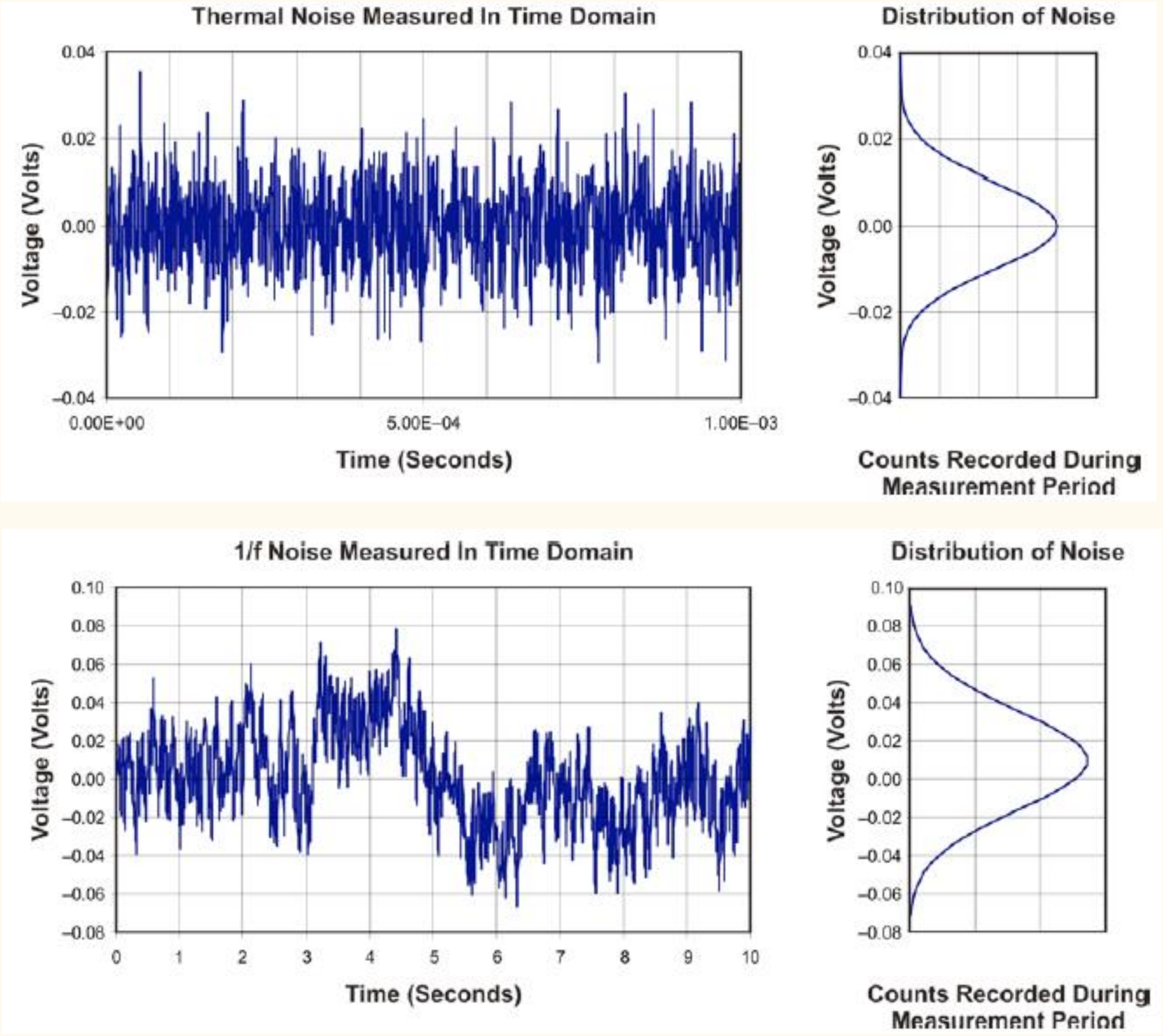
Two-level system



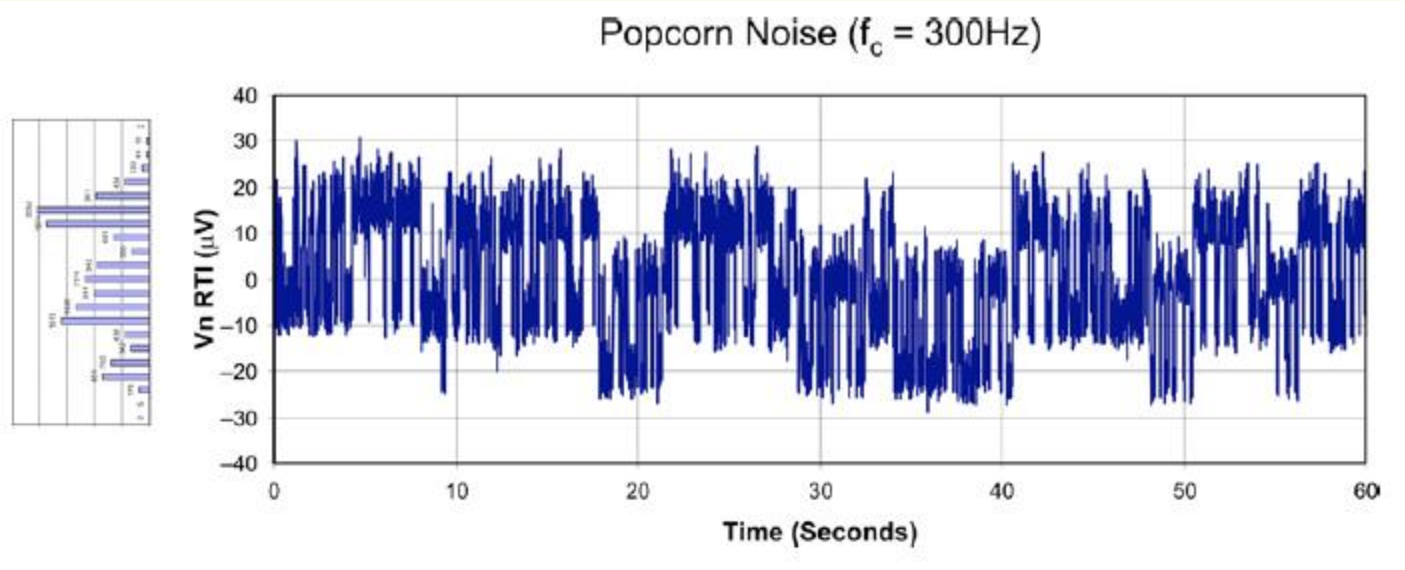


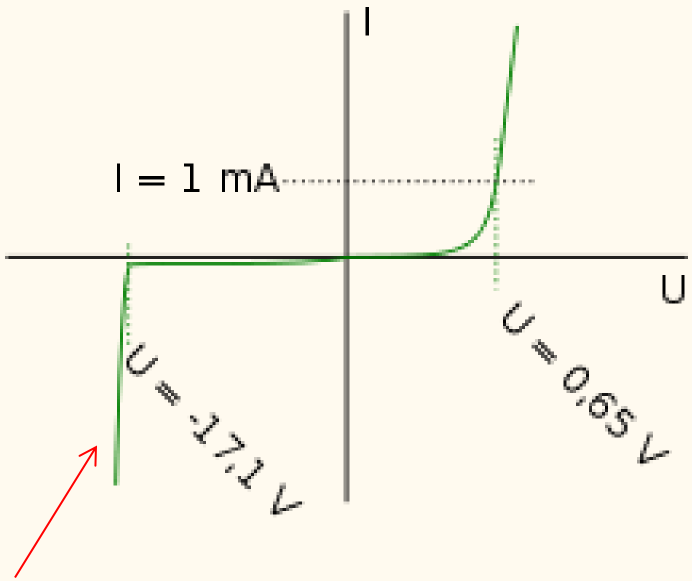
# Amplitude distributions of random-type noises



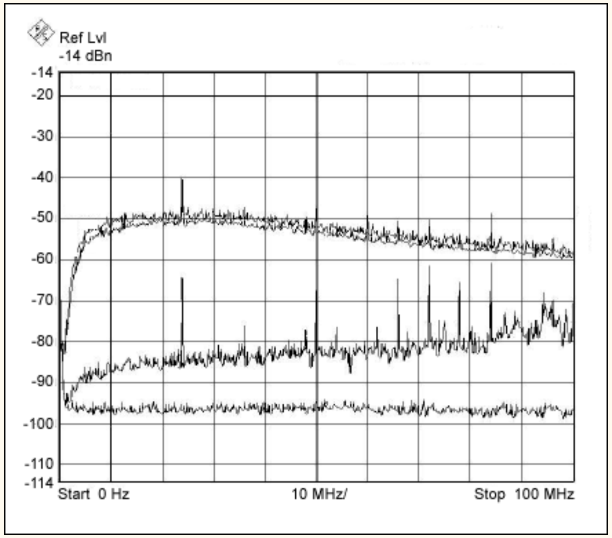
# Amplitude distribution of popcorn noise



# Avalanche noise



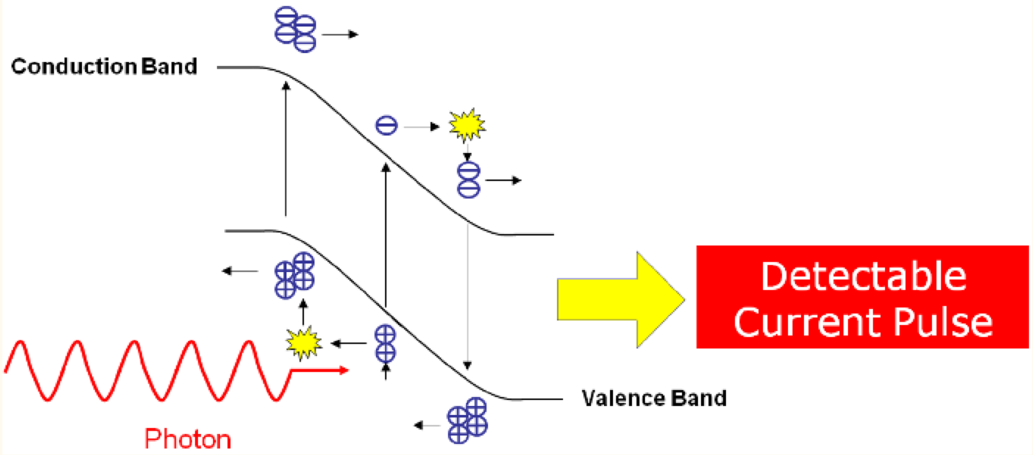

avalanche or Zener breakdown




white noise

Avalanche Photo-Diode (APD)

Zener voltage standard diode



# 6.2 Noises from Amplifiers

Amplifiers: the elements have characteristic noises,
power sources work as noise sources

→ Noiseless amplifier + Noise source = Amplifier with noise

Power gain $G_p$

$$e^2_{\mathrm{intotal}} = j_n^2 R^2 + e_{\mathrm{R}}^2 + e_a^2 = e^2_{\mathrm{out}}/G_{\mathrm{p}}$$

Signal to noise ratio: S/N ratio

Noise Figure: $\mathrm{NF} = 10\log_{10}\dfrac{(S/N)_{\mathrm{in}}}{(S/N)_{\mathrm{out}}} = 10\log_{10}\dfrac{S_{\mathrm{in}}N_{\mathrm{out}}}{S_{\mathrm{out}}N_{\mathrm{in}}}$

$$N_{\mathrm{out}} = G_{\mathrm{p}}\overline{e_{\mathrm{N}}^2}$$

$$\mathrm{NF} = 10\log_{10}\frac{S_{\mathrm{in}}G_{\mathrm{p}}\overline{e_{\mathrm{N}}^2}}{S_{\mathrm{in}}G_{\mathrm{p}}\overline{e_{\mathrm{R}}^2}} = 10\log_{10}\frac{\overline{e_{\mathrm{N}}^2}}{\overline{e_{\mathrm{R}}^2}} = 10\log_{10}\frac{\overline{e_n^2} + \overline{e_{\mathrm{R}}^2} + \overline{j_n^2}R^2}{\overline{e_{\mathrm{R}}^2}}$$

# 6.2.2 Noise impedance matching

Noise temperature and matched source impedance

$$T_{\mathrm{a}} = \frac{\sqrt{\overline{e_n^2}\;\overline{j_n^2}}}{2k_{\mathrm{B}}}, \qquad R_{\mathrm{bs}} = \sqrt{\frac{\overline{e_n^2}}{\overline{j_n^2}}}$$

Output noise temperature:

$$T_{\mathrm{n}} = \left(1 + \frac{\mathrm{Re}(1/Z_{\mathrm{i}})}{\mathrm{Re}(1/Z_{\mathrm{s}})}\right) \frac{T_{\mathrm{a}}}{2\mathrm{Re}Z} \left(\frac{|Z|^2}{R_{\mathrm{bs}}} + R_{\mathrm{bs}}\right), \qquad \frac{1}{Z} \equiv \frac{1}{Z_{\mathrm{i}}} + \frac{1}{Z_{\mathrm{s}}}$$

Minimize $T_n$: $\quad Z_{\mathrm{i}} = \dfrac{1}{R_{\mathrm{bs}}^{-1} - Z_{\mathrm{s}}^{-1}}$ Noise matching condition

$$T_{\mathrm{n}} = \left(1 + \frac{\mathrm{Re}(1/Z_{\mathrm{i}})}{\mathrm{Re}(1/Z_{\mathrm{s}})}\right) T_{\mathrm{a}}$$

# References

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Anton F. P. van Putten, "Electronic Measurement Systems",
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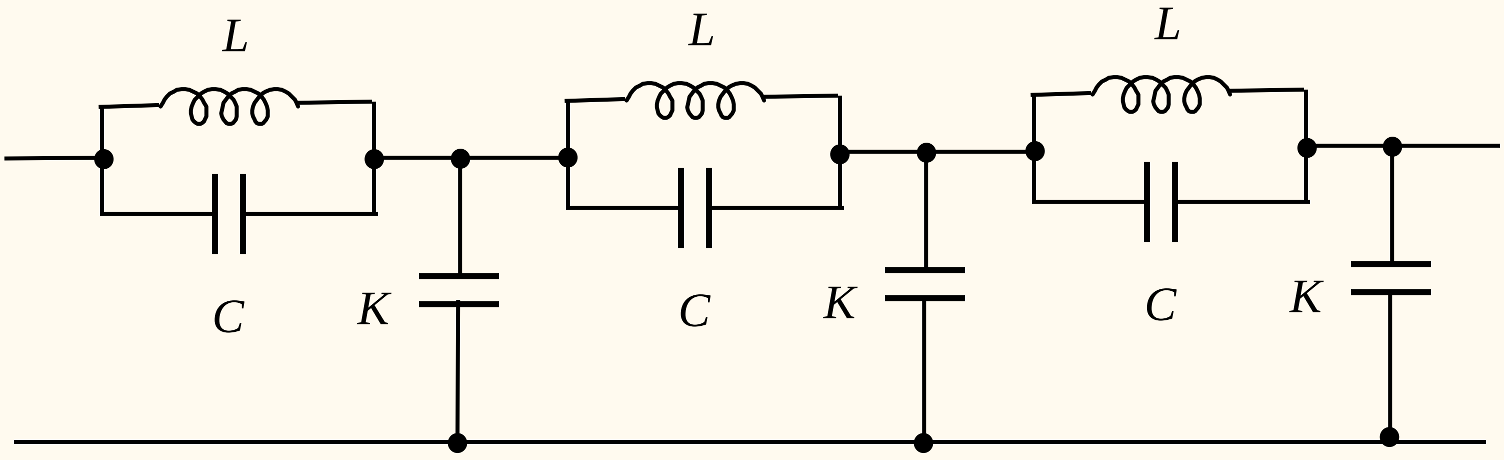
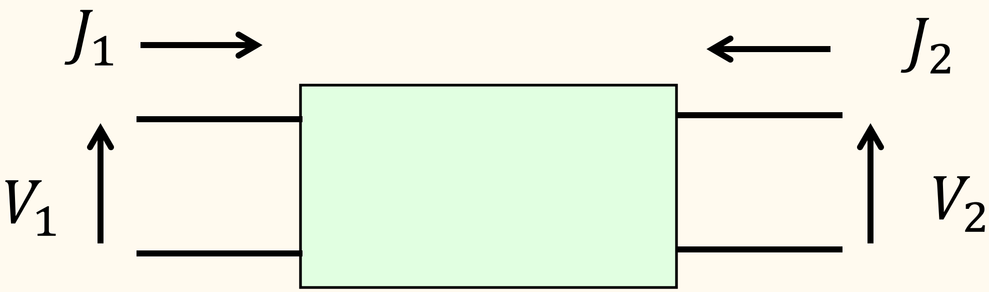
# Exercise 5-1

Obtain the dispersion relation in the following transmission line.



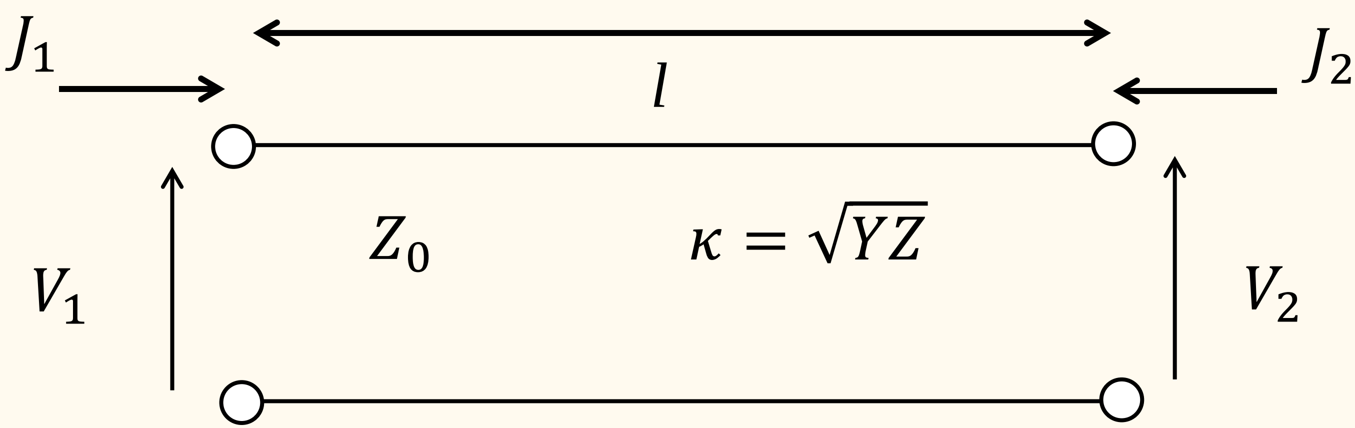
# Exercise 5-2




$$\begin{pmatrix} V_1 \\ J_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_2 \\ J_2 \end{pmatrix}$$

Remember F-matrix (cascade matrix) defined above.

Write down the F-matrix form of the transmission line shown below.



# Exercise 5-3

Show that the power spectrum $G(f)$ of voltage noise across the impedance

$$Z(f) = R(f) + iY(f)$$

is given as

$$G(f) = 4R(f)k_{\mathrm{B}}T.$$