

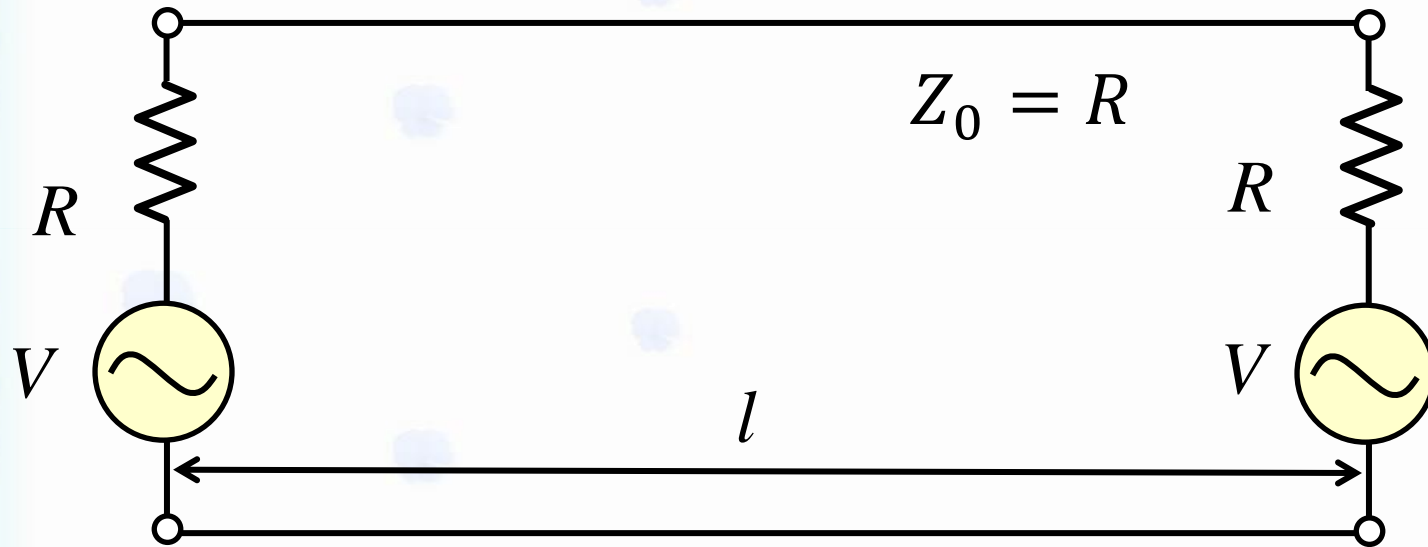
# Electric Circuits for Physicists

## 電子回路論第11回

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# Review: Nyquist theorem



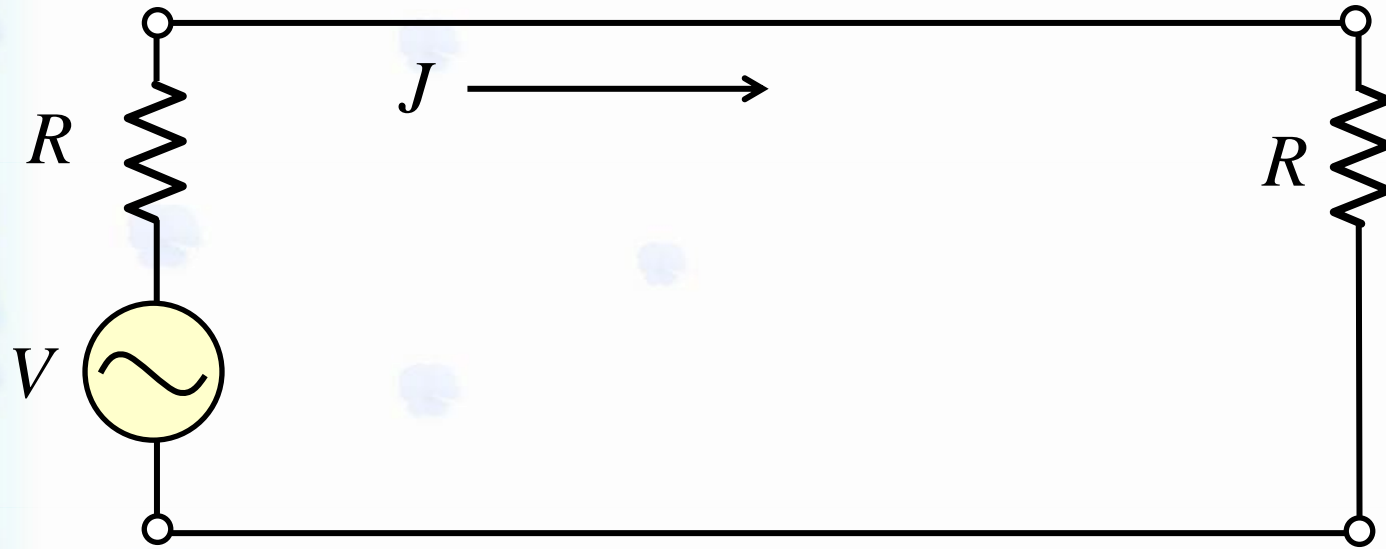
Thermal energy density per freedom

$$\frac{\hbar\omega}{\exp(\hbar\omega/k_{\text{B}}T) - 1} \sim \frac{\hbar\omega}{1 + (\hbar\omega/k_{\text{B}}T) - 1} = k_{\text{B}}T \quad (k_{\text{B}}T \gg \hbar\omega)$$

Thermal energy density in band  $\Delta\omega$

$$2 \frac{\Delta\omega}{\delta\omega} k_{\text{B}}T = \frac{2k_{\text{B}}Tl}{2\pi c^*} \Delta\omega, \text{ a half of which flows in one-direction}$$

# Review: Nyquist theorem



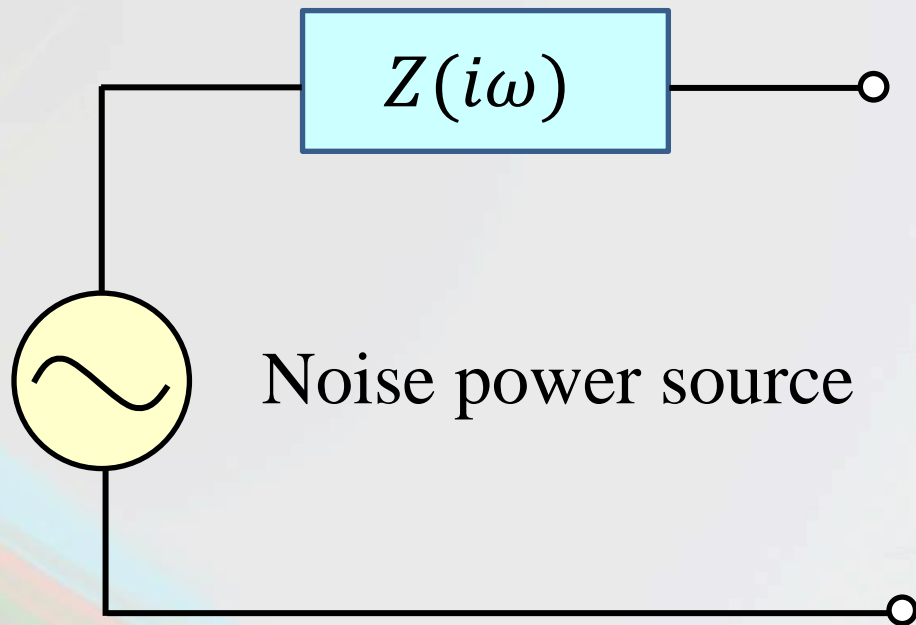
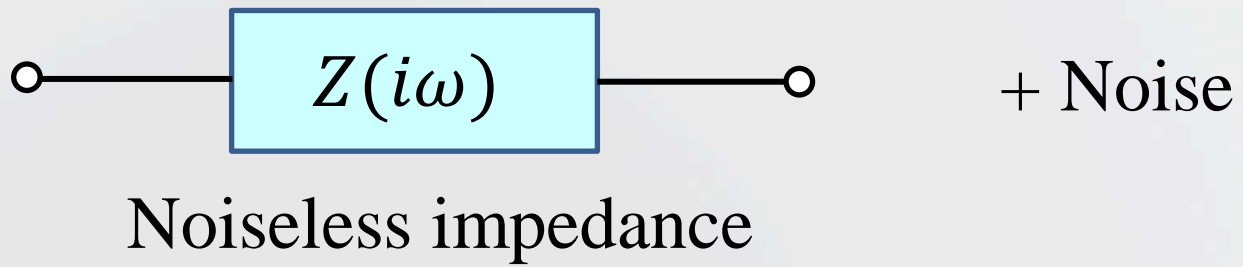
Energy flowing out from the end:

$$\frac{k_B T l}{2\pi c^*} \Delta\omega \times \frac{1}{l} \times c^* = k_B T \Delta f \quad (2\pi f = \omega)$$

equals the energy supplied from the noise source.

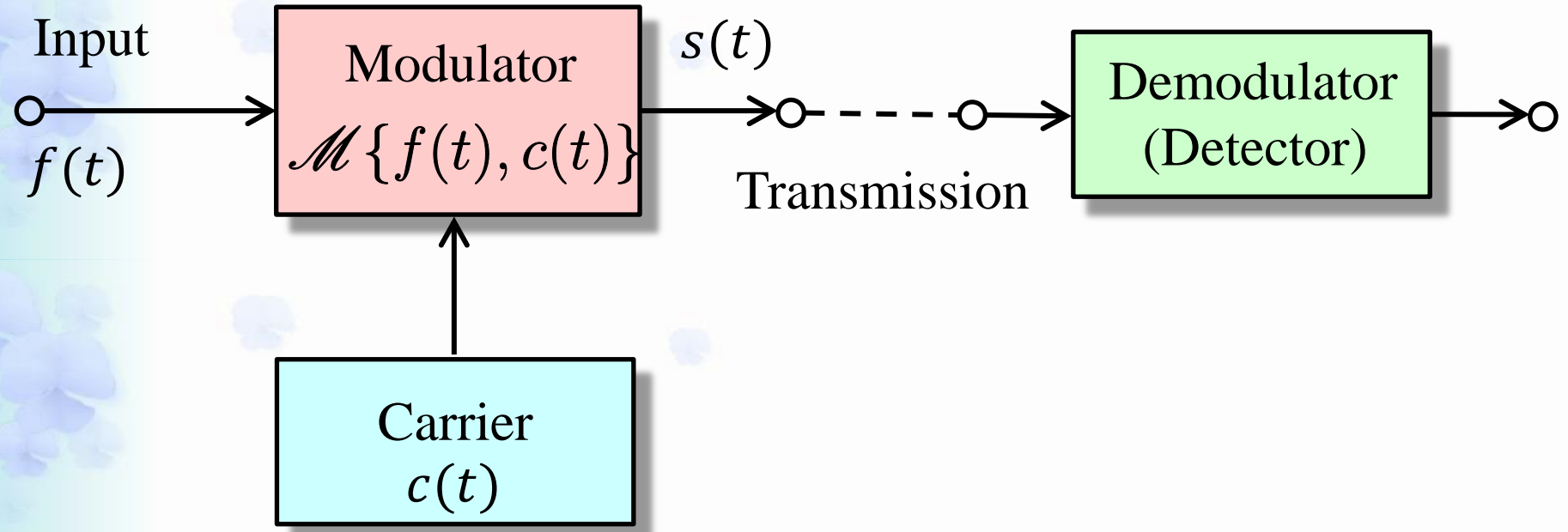
$$\overline{J^2} R = k_B T \Delta f, \quad \overline{V^2} = 4Rk_B T \Delta f$$

# Electric circuit treatment of noise



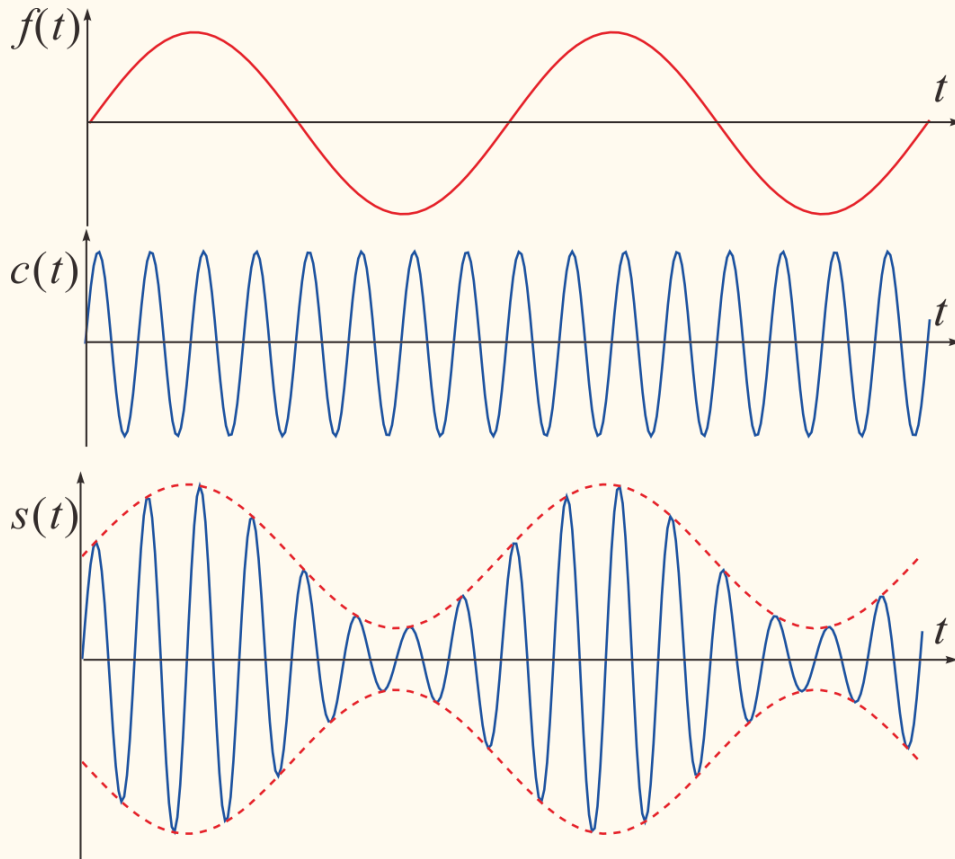
# 6.3 Signal transmission

Electric communication {  
Baseband communication  
Carrier communication



Modulation {  
Amplitude modulation  
Frequency (Phase) modulation } Analog  
Pulse

## 6.3.2 Amplitude modulation



$$c(t) = A \cos \omega_c t$$

$$s(t) = A[1 + m f(t)] \cos \omega_c t$$

$m$ : Modulation index

$$0 < m \leq 1$$

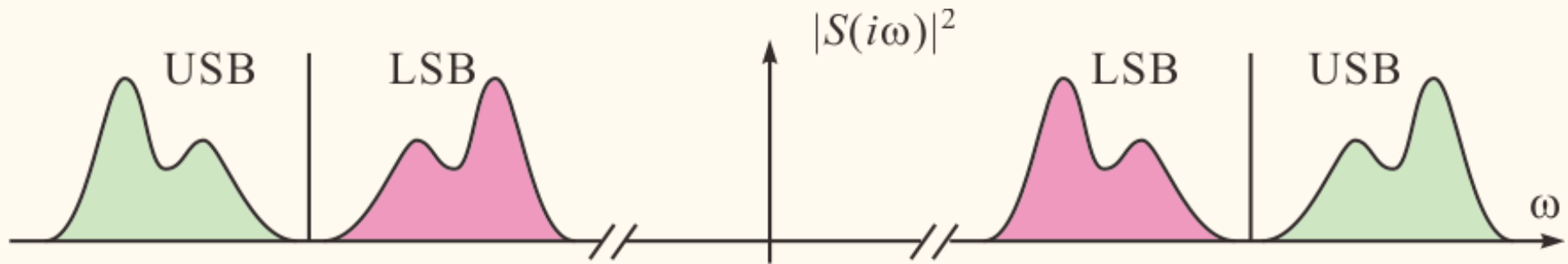
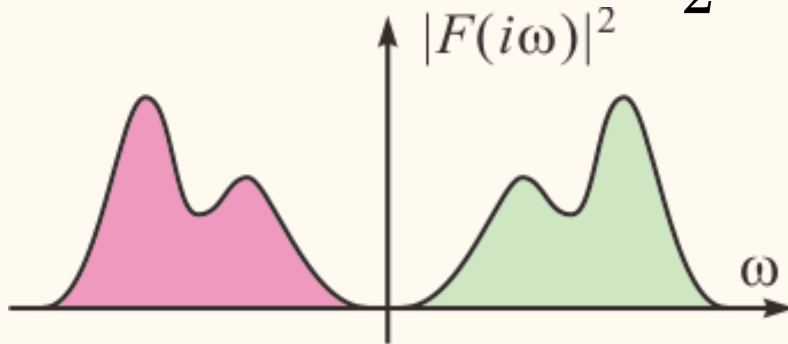
$$\begin{aligned} S(i\omega) &= \int_{-\infty}^{\infty} s(t) e^{i\omega t} dt = \int_{-\infty}^{\infty} A[1 + m f(t)] \cos(\omega_c t) e^{i\omega t} dt \\ &= A \left\{ \pi [\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] \right. \\ &\quad \left. + \frac{m}{2} [F(i(\omega - \omega_c)) + F(i(\omega + \omega_c))] \right\} \end{aligned}$$

## 6.3.2 Amplitude modulation

$$\begin{aligned} S(i\omega) &= \int_{-\infty}^{\infty} s(t)e^{i\omega t} dt = \int_{-\infty}^{\infty} A[i + mf(t)] \cos(\omega_c t)e^{i\omega t} dt \\ &= A \left\{ \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] \right. \\ &\quad \left. + \frac{m}{2}[F(i(\omega - \omega_c)) + F(i(\omega + \omega_c)))] \right\} \end{aligned}$$

$$F(i\omega) = \mathcal{F}\{f(t)\}$$

$f(t)$ : Real

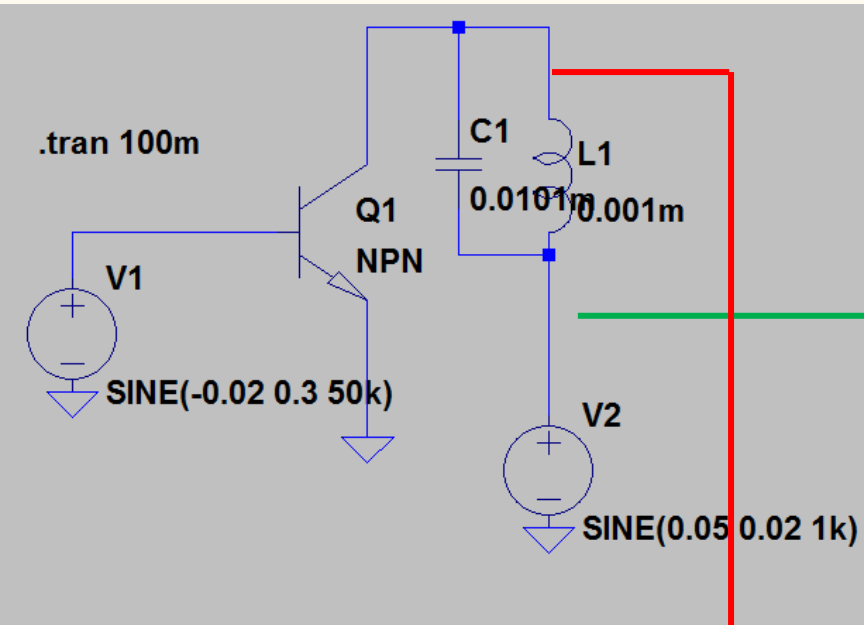


Upper side band (USB), Lower side band (LSB)

# 6.3.2 Amplitude modulation (circuit example)

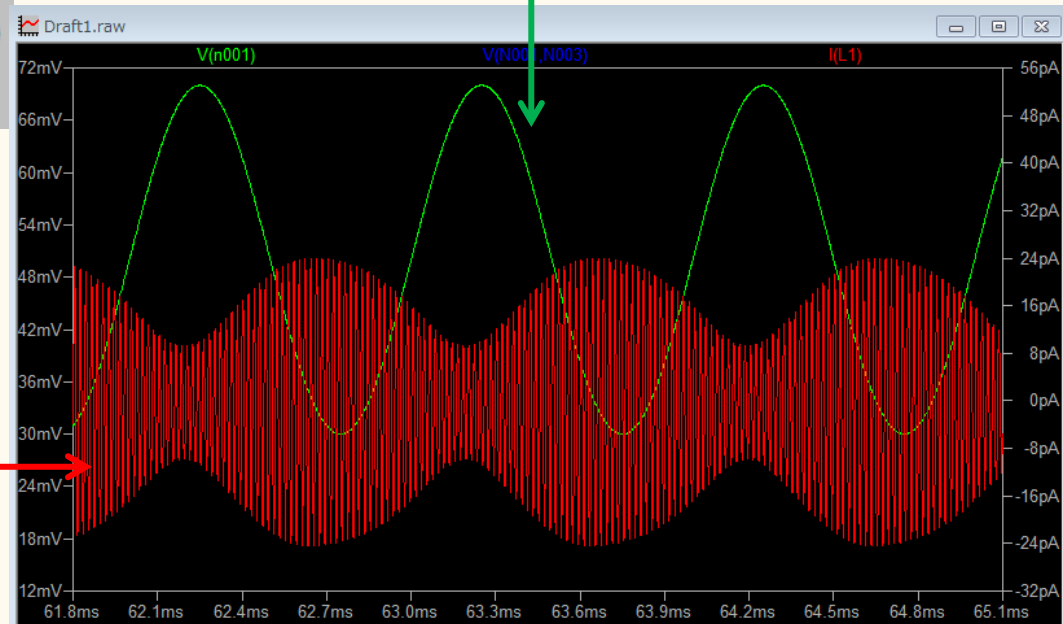
## Collector modulation circuit

## C-class amplification (non-linear) region



Modulation voltage

Current through the inductor

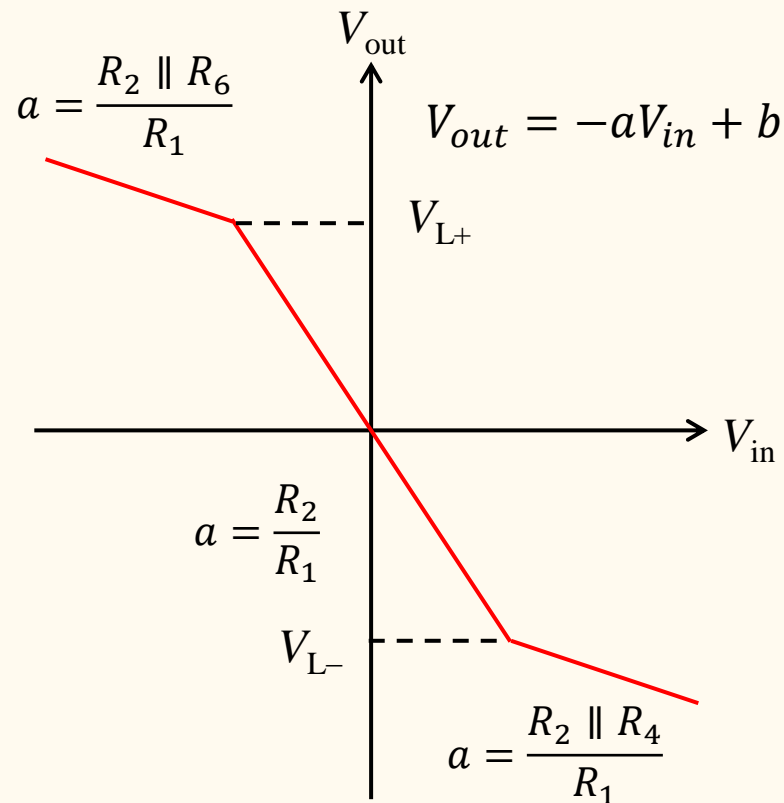
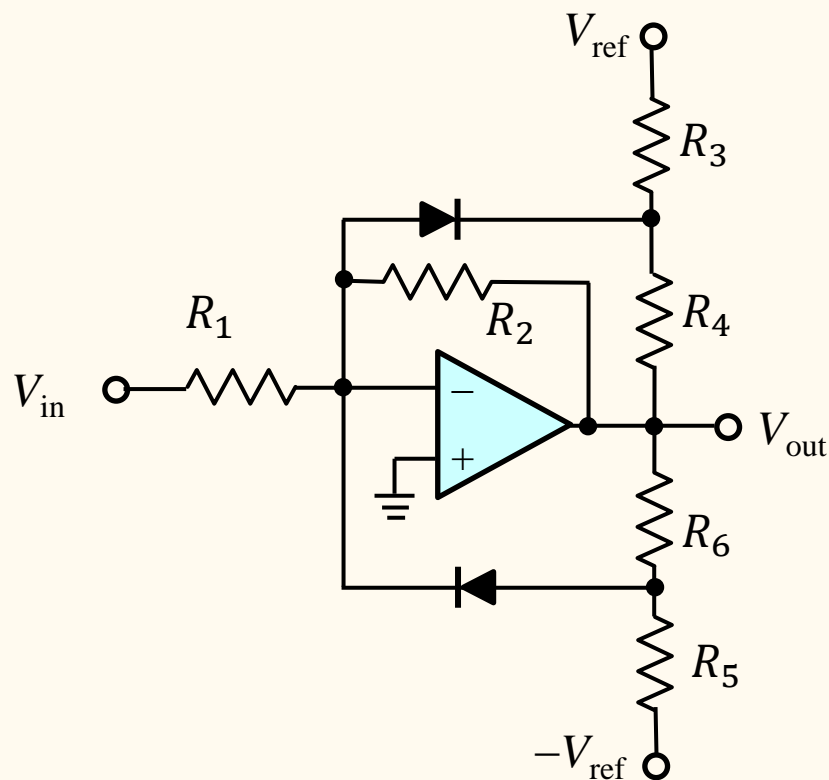




# 6.3.2 Amplitude modulation (circuit example2)

## Modulation of oscillator circuit

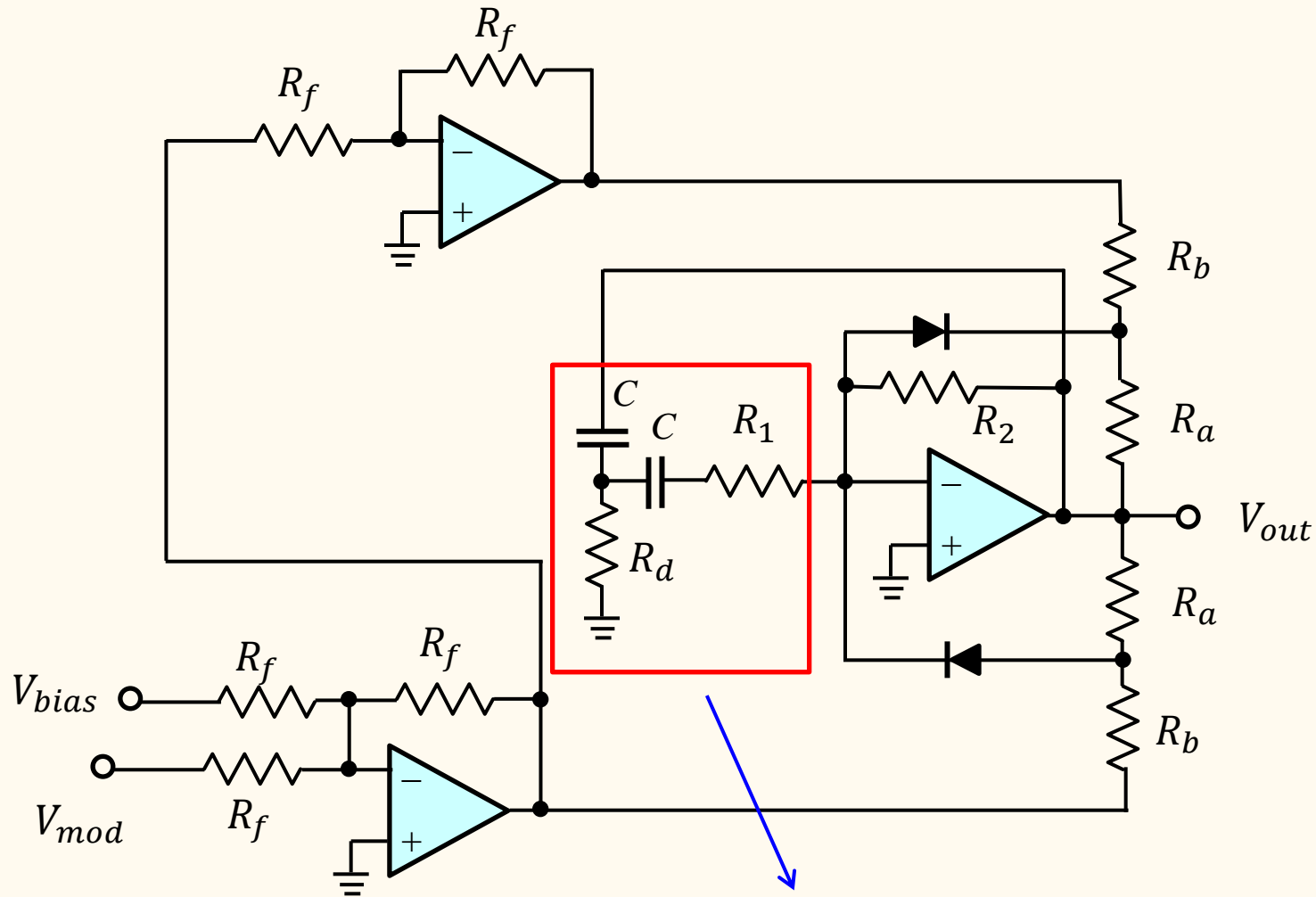
### Soft limiter circuit



$$V_{L-} = -\frac{R_4}{R_3} V_{ref} - \left(1 + \frac{R_4}{R_3}\right) V_{th}$$

: controllable with  $V_{ref}$

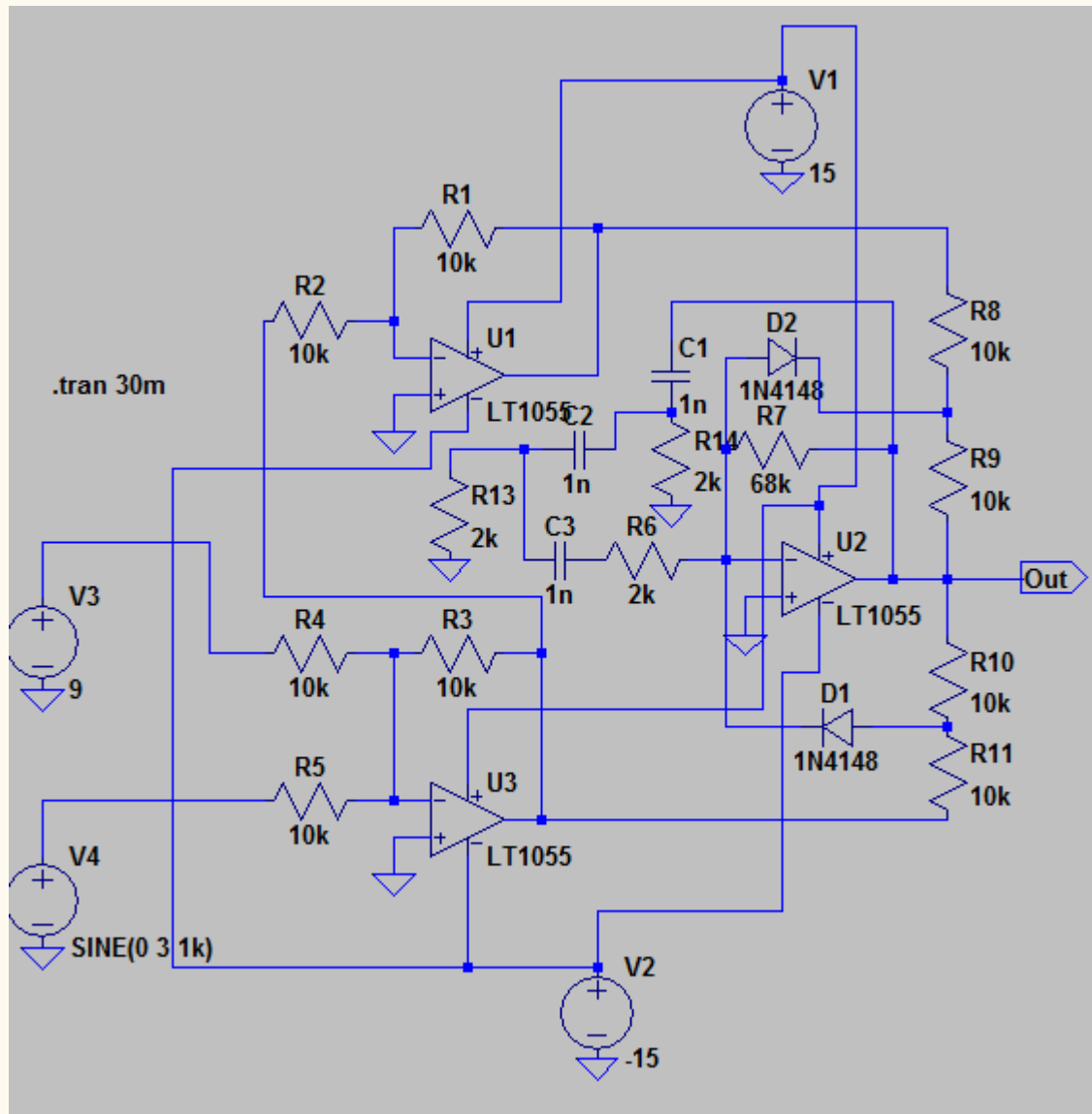
## 6.3.2 Amplitude modulation (circuit example2)



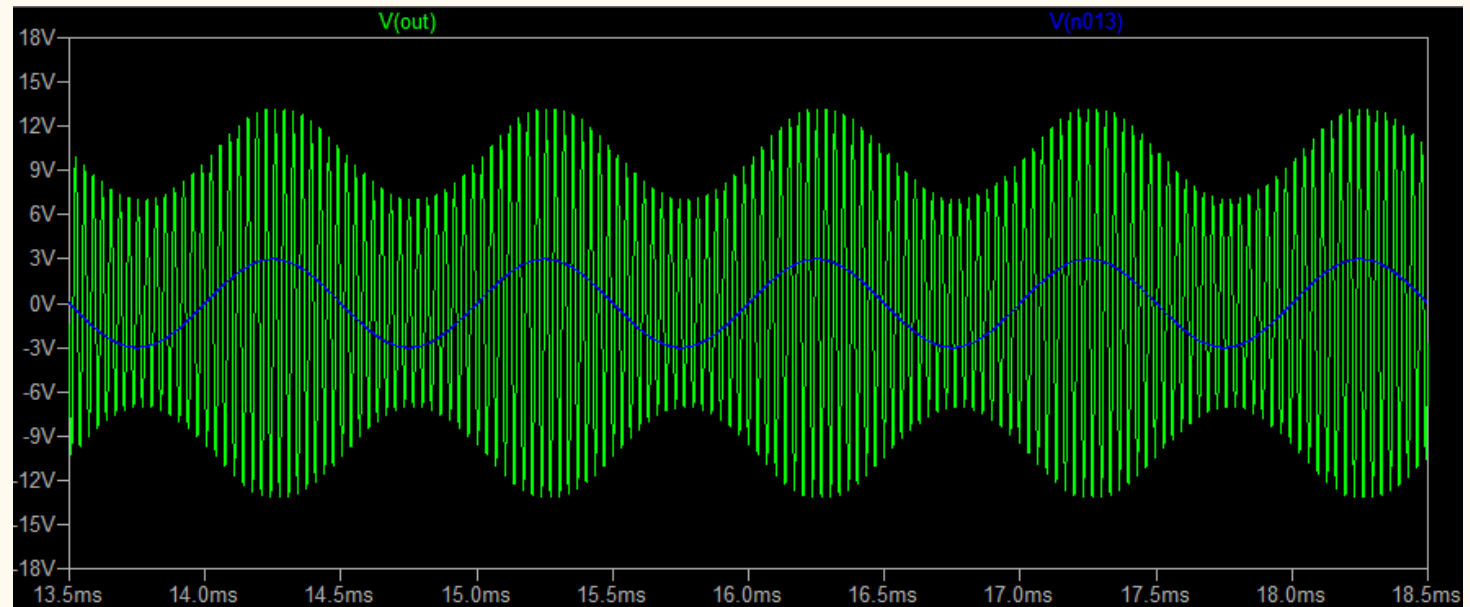
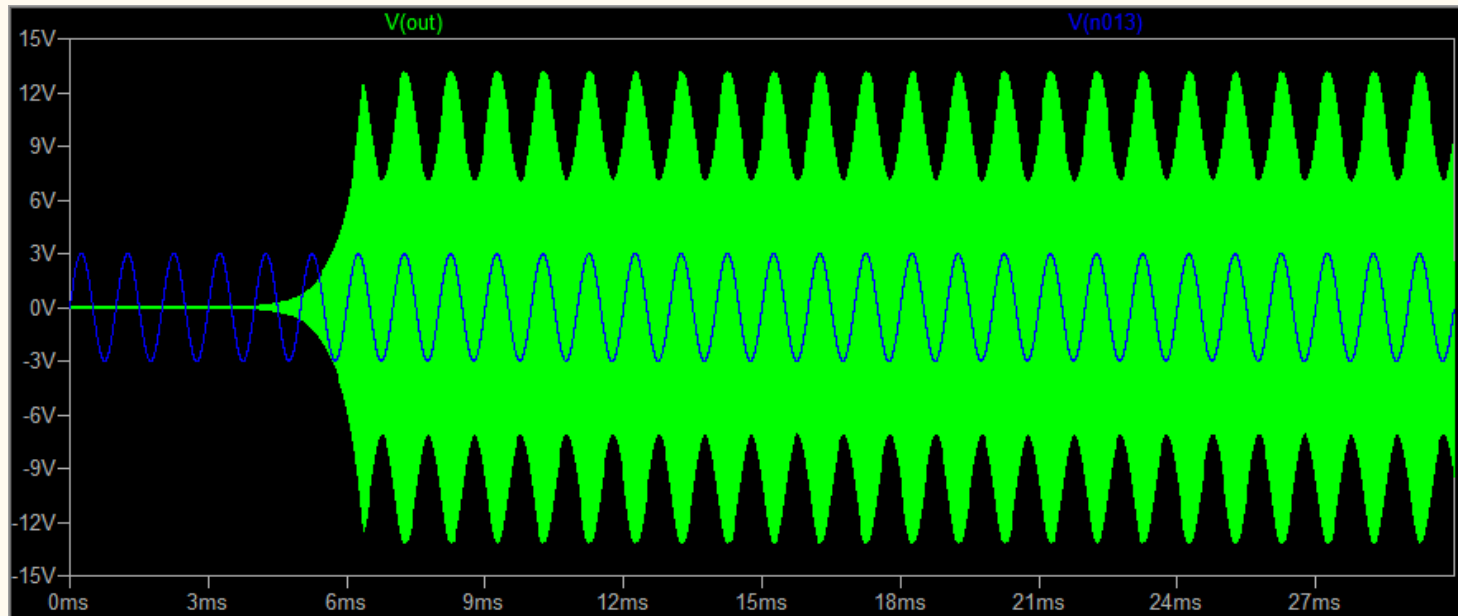
Delayed feedback → Oscillation

The amplitude is softly limited with the modulation voltage.

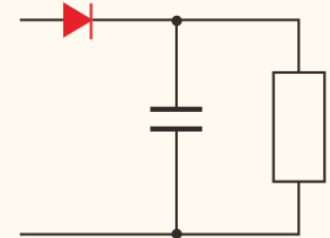
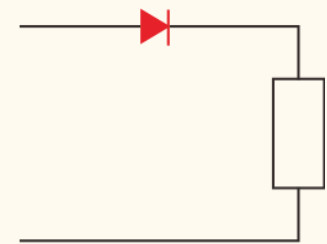
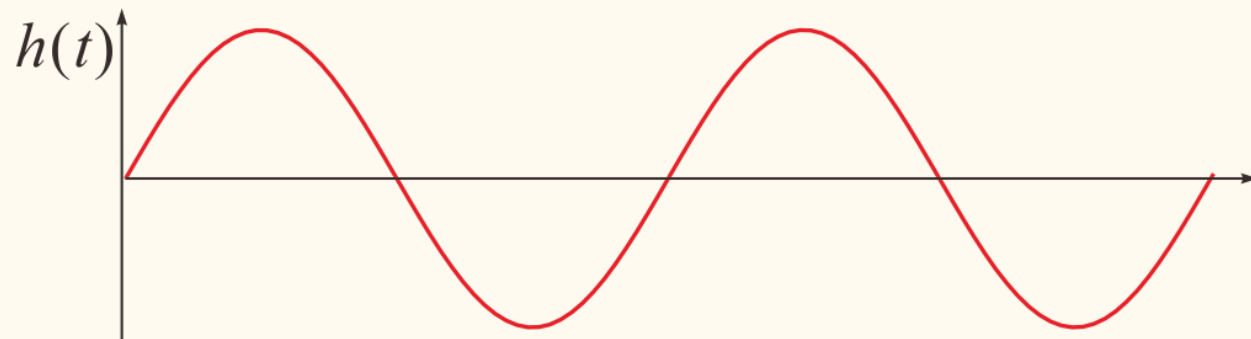
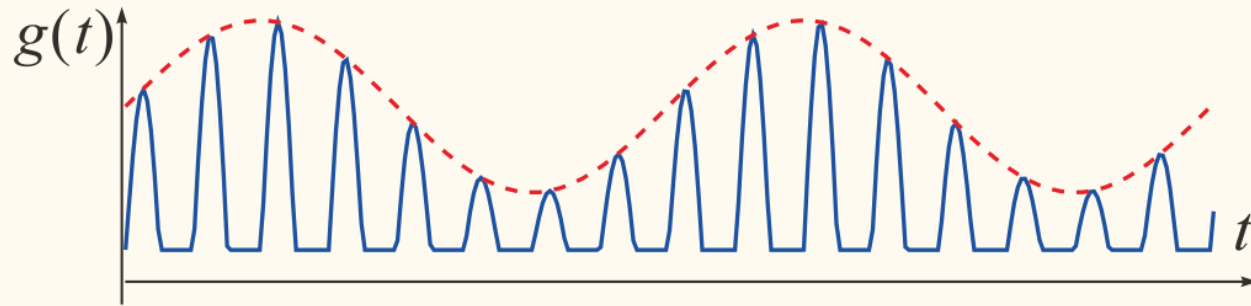
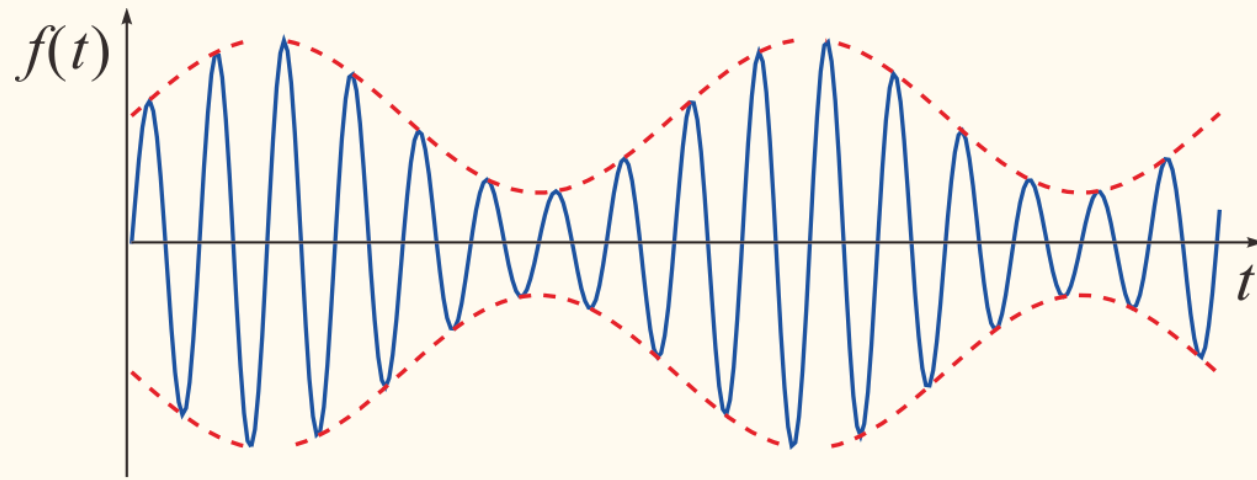
## 6.3.2 Amplitude modulation (circuit example2)



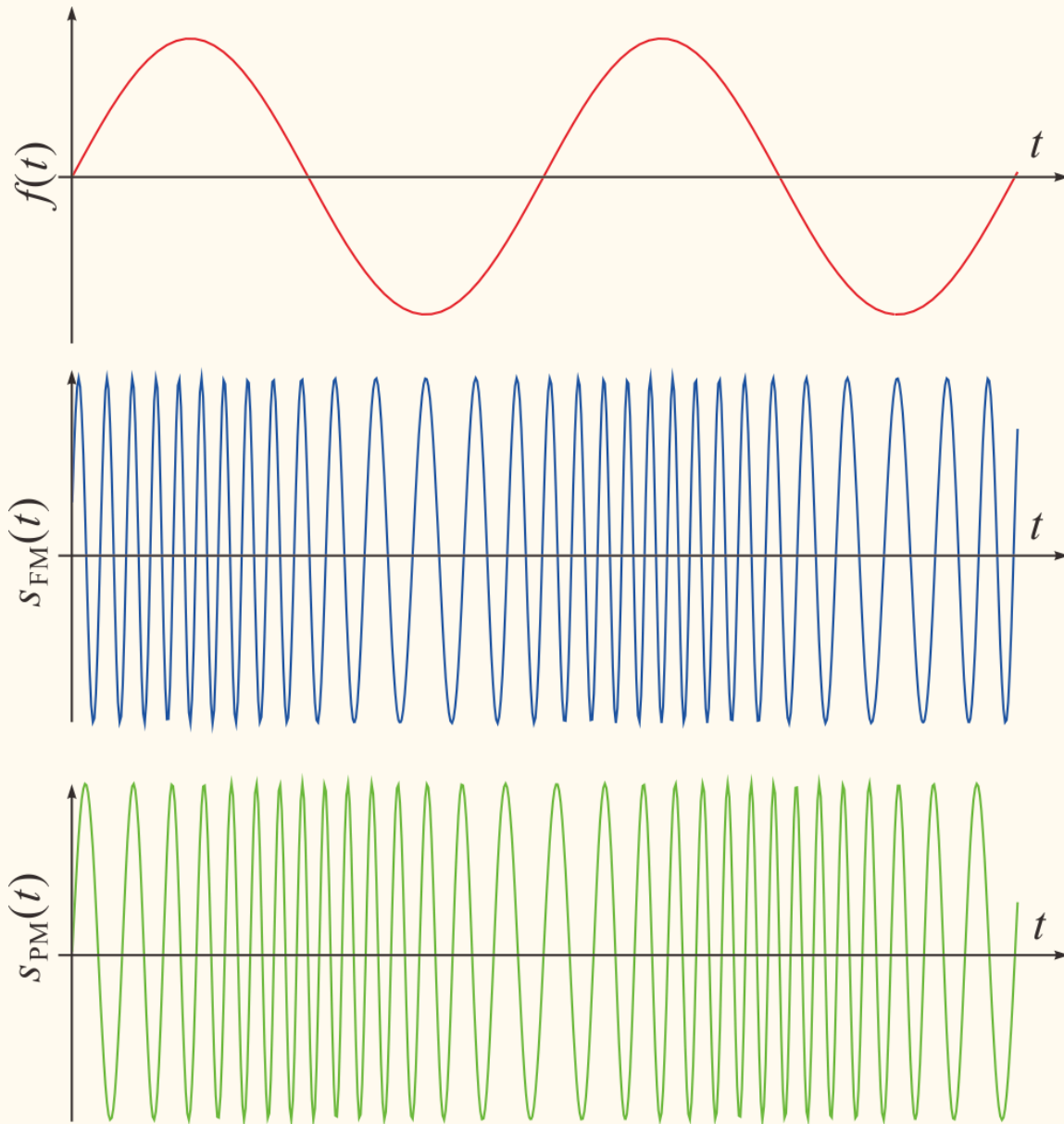
## 6.3.2 Amplitude modulation (circuit example2)



## 6.3.2 Amplitude modulation (demodulation)



## 6.3.3 Angle modulation



## 6.3.3 Angle modulation

$$s(t) = A \cos \theta_i(t), \quad \theta_i(t) = \omega_c t + \phi[t, f(t)]$$

Differential angular frequency  $\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_c + \frac{d\phi[t, f(t)]}{dt}$

$$\frac{d\phi[t, f(t)]}{dt} = k_f f(t) \quad (\text{Frequency Modulation, FM}),$$

$$\phi[t, f(t)] = k_p f(t) \quad (\text{Phase Modulation, PM})$$

$$s_{\text{FM}}(t) = A \cos \left[ \omega_c t + k_f \int^t f(\tau) d\tau \right],$$

$$s_{\text{PM}}(t) = A \cos[\omega_c t + k_p f(t)]$$

Frequency  $\omega$  component: only phase shift  $\pi/2$  :

No difference in signal outlook.

### 6.3.3 Angle modulation (Frequency modulation)

$$f(t) = A_p \cos \omega_p t$$

$$s_{\text{FM}} = A \cos(\omega_c t + \beta \sin \omega_p t) = A \operatorname{Re} [\exp(i\omega_c t) \exp(i\beta \sin \omega_p t)]$$

$$\left( \beta \equiv \frac{k_f A_p}{\omega_p} = \frac{\Delta f}{f_p} \right)$$

$\sin \omega_p t$  : Periodic function with  $T = 2\pi/\omega_p$

$$\exp(i\beta \sin \omega_p t) = \sum_{n=-\infty}^{\infty} c_n \exp(in\omega_p t), \quad \text{Fourier series expansion}$$

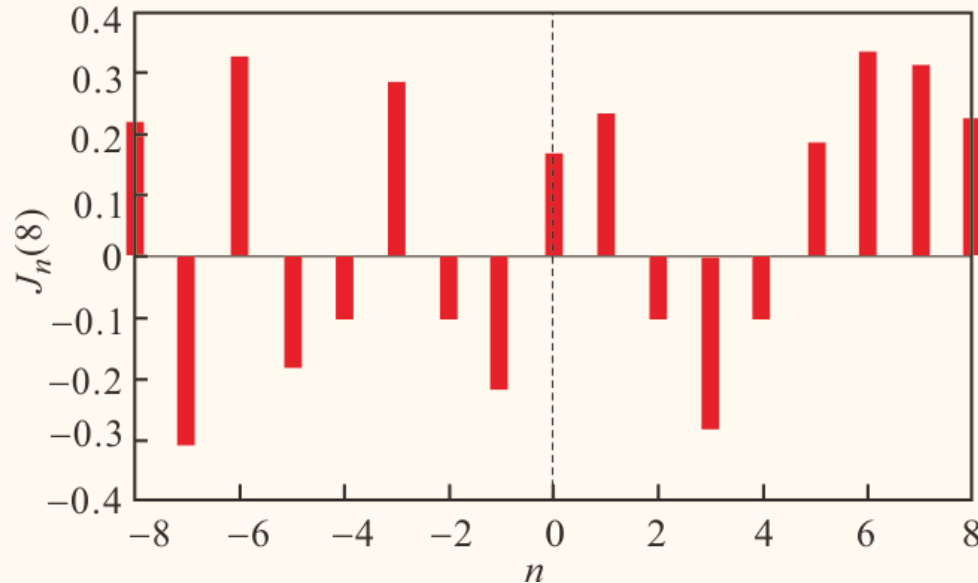
$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \exp(i\beta \sin \omega_p t) \exp(-in\omega_p t) dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[i(\beta \sin \theta - n\theta)] d\theta = J_n(\beta)$$

First kind Bessel function



## 6.3.3 Angle modulation (Frequency modulation)



$$s_{\text{FM}}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[(\omega_c + n\omega_p)t]$$

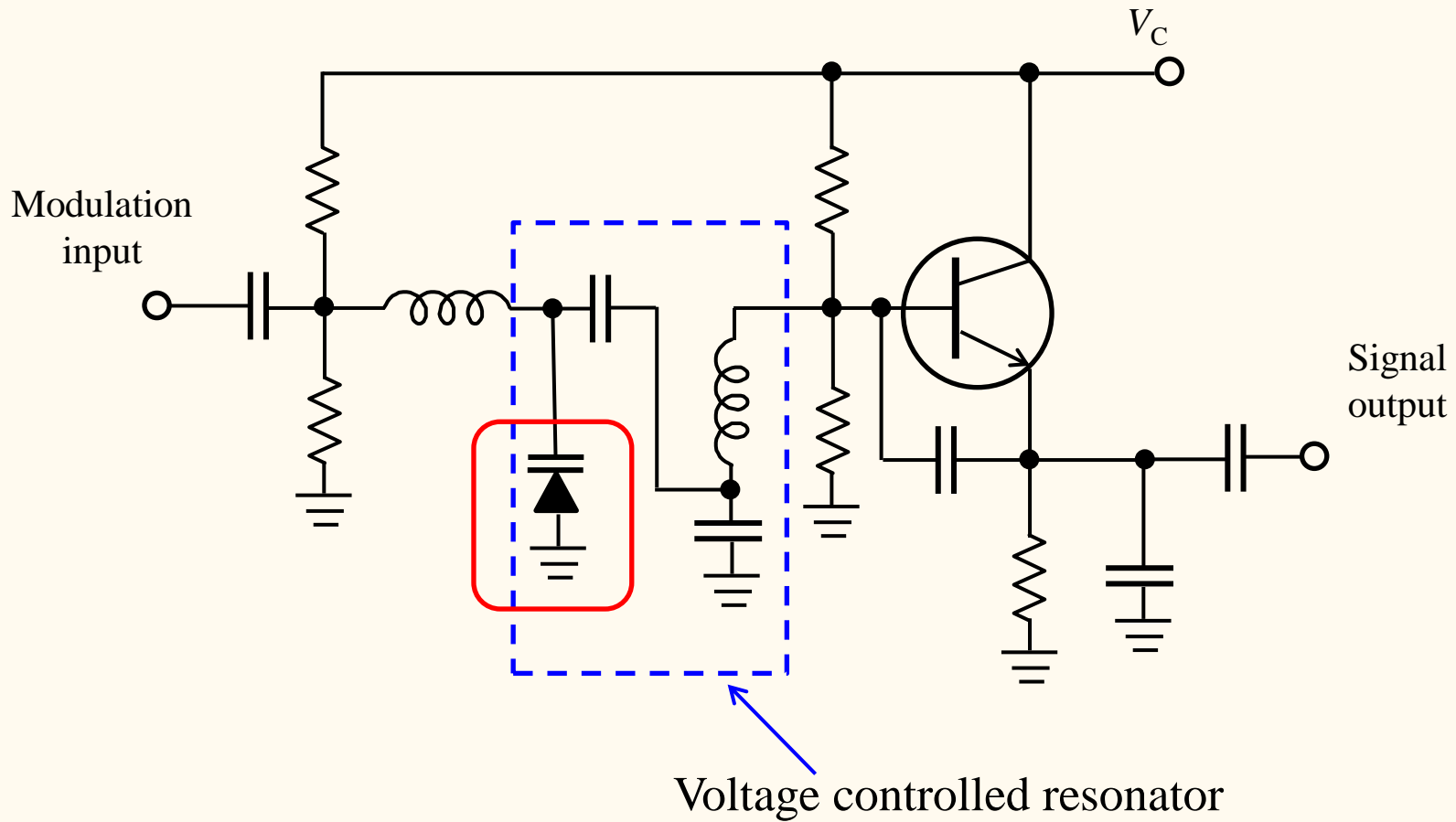
$$S_{\text{FM}}(i\omega) = \pi A \sum_{n=-\infty}^{\infty} J_n(\beta) \{ \delta[\omega - (\omega_c + n\omega_p)] + \delta[\omega + (\omega_c + n\omega_p)] \}$$

Actual band width:

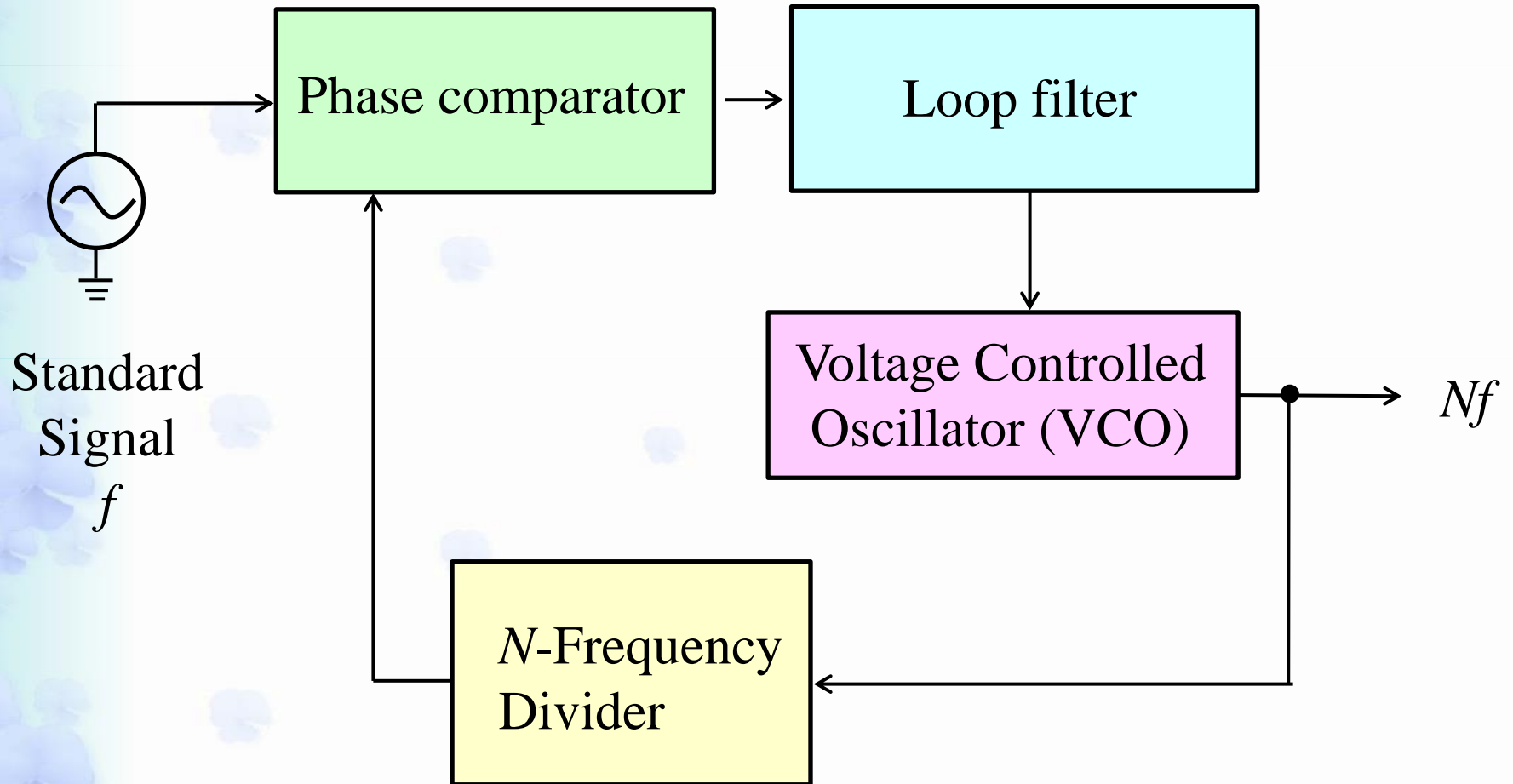
$$\omega_{\text{bw}} = 2(\omega_f + \xi\omega_w) \quad 1 \leq \xi \leq 2$$

## 6.3.3 Angle modulation (circuit example)

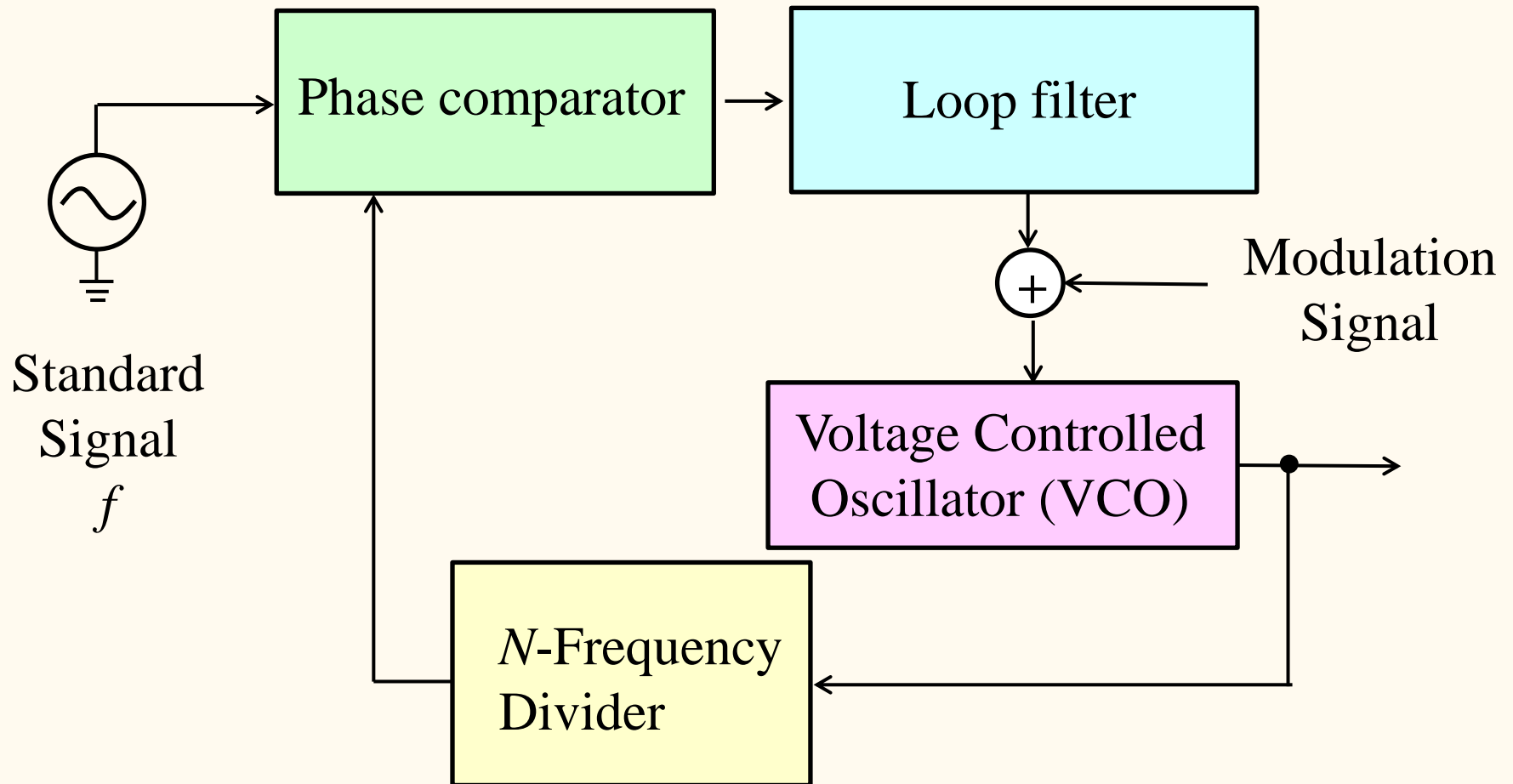
### Voltage Controlled Oscillator (VCO)



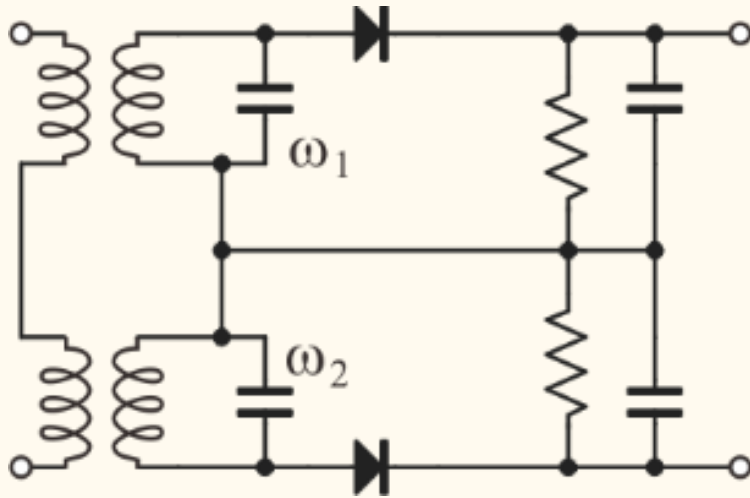
# Phase Lock Loop (PLL)



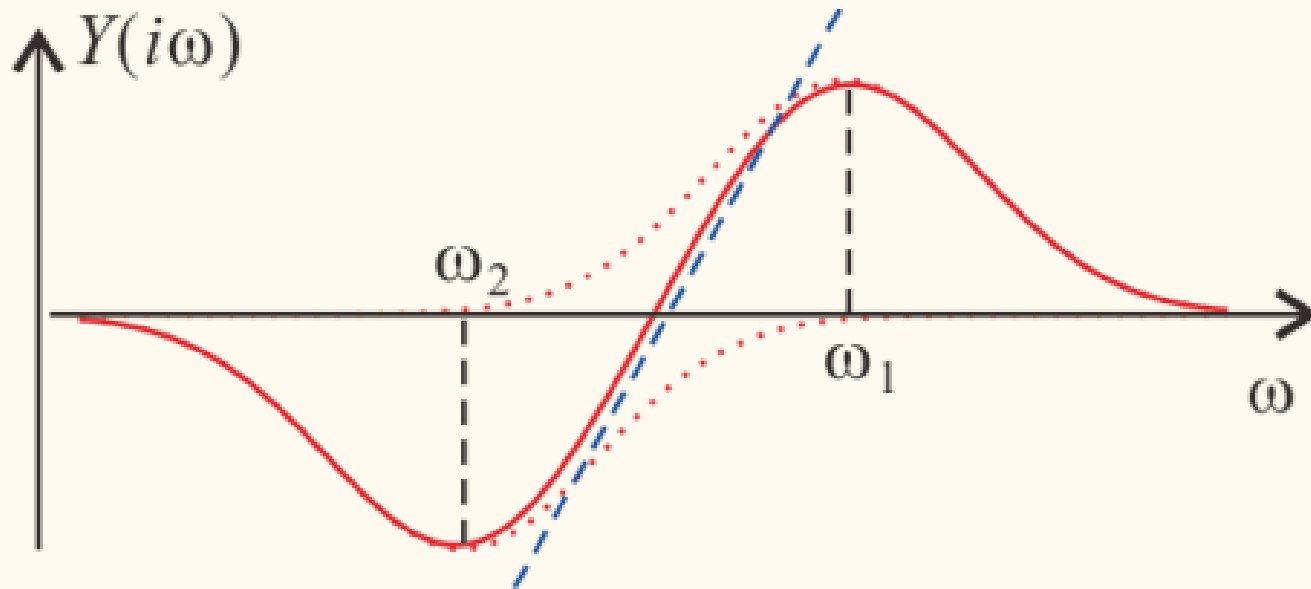
## 6.3.3 Angle modulation (circuit example)



## 6.3.4 Angle modulation (Frequency demodulation)

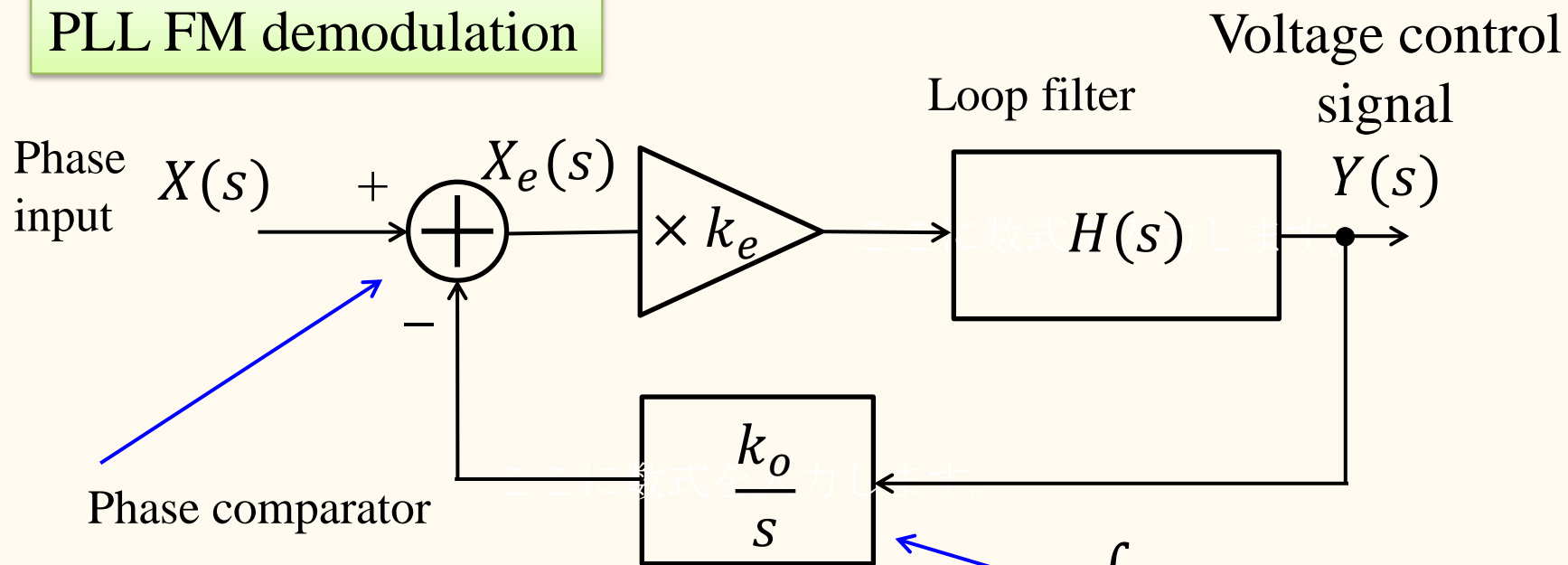


Double tuned circuit



# 6.3.4 Angle modulation (frequency demodulation)

## PLL FM demodulation



$$Y(s) = \frac{sk_e H(s)}{s + k_e k_o H(s)} X(s)$$

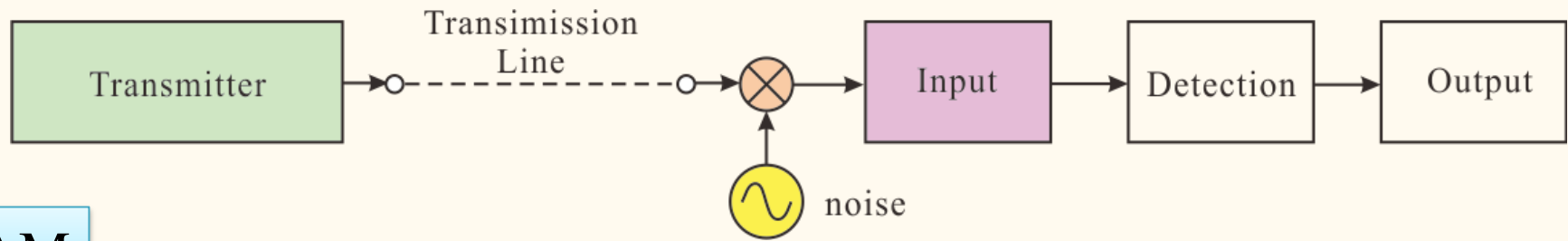
$k_o \int y(t) dt$

$g(t)$ : Frequency modulation signal (original)

$$\phi(t) = k_f \int_{-\infty}^t g(\tau) d\tau, \quad sX(s) = k_f G(s)$$

$$\therefore Y(s) = \frac{k_f k_e H(s)}{s + k_e k_o H(s)} G(s) \approx \frac{k_f}{k_o} G(s)$$

# 6.3.5 Modulation and noise



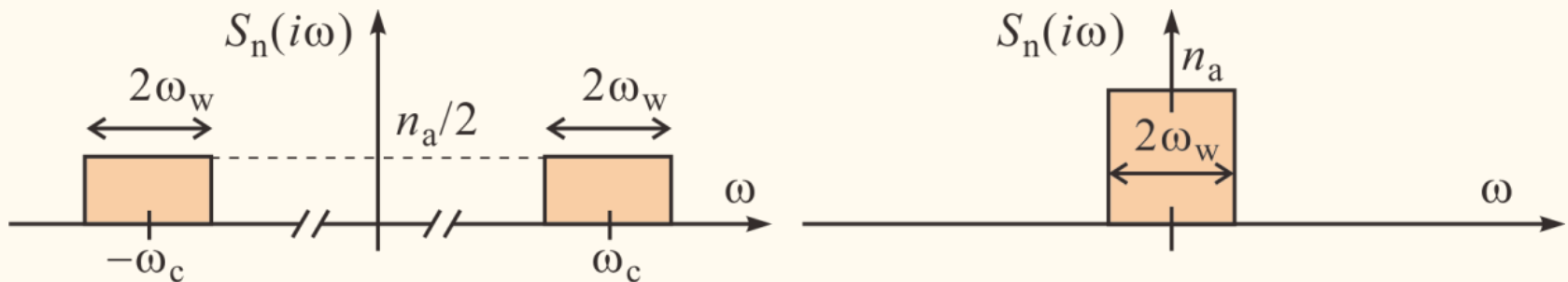
AM

Received signal  $r(t) = A_r [1 + m f(t)] \cos \omega_c t + n_i(t)$

Demodulated output  $g(t) = A_r m f(t) + n_o(t)$

Averaged signal power

received:  $S_{pr} = \frac{A_r^2}{2} + \frac{(A_r m)^2}{2} \langle f^2 \rangle$ ,    output:  $S_{po} = A_r^2 m^2 \langle f^2 \rangle$



Received

Demodulated

## 6.3.5 Modulation and noise

$\omega_W$  : Noise bandwidth (assumption: white)

$$\text{Noise power: } \underbrace{2 \times \frac{n_a}{2 \times 2\pi} \times 2\omega_W = \frac{n_a \omega_W}{\pi}}_{\text{Received}}, \quad \underbrace{\frac{n_a \times 2\omega_W}{2\pi} = \frac{n_a \omega_W}{\pi}}_{\text{Demodulated}}$$

$$\left. \frac{S}{N} \right|_{\text{in}} = \frac{\pi[A_r^2 + (A_r m)^2 \langle f^2 \rangle]}{2n_a \omega_W}, \quad \left. \frac{S}{N} \right|_{\text{out}} = \frac{\pi A_r^2 m^2 \langle f^2 \rangle}{n_a \omega_W} = 2\eta \left. \frac{S}{N} \right|_{\text{in}}$$

$$\eta = \frac{m^2 \langle f^2 \rangle}{1 + m^2 \langle f^2 \rangle} \quad \text{:Power transmission efficiency}$$

$$0 < m \leq 1 \rightarrow \eta < \frac{1}{2}$$

$$\text{Input sinusoidal: } \langle f^2 \rangle = \frac{1}{2} \rightarrow \eta < \frac{1}{3}$$



## 6.3.5 Modulation and noise

FM, PM

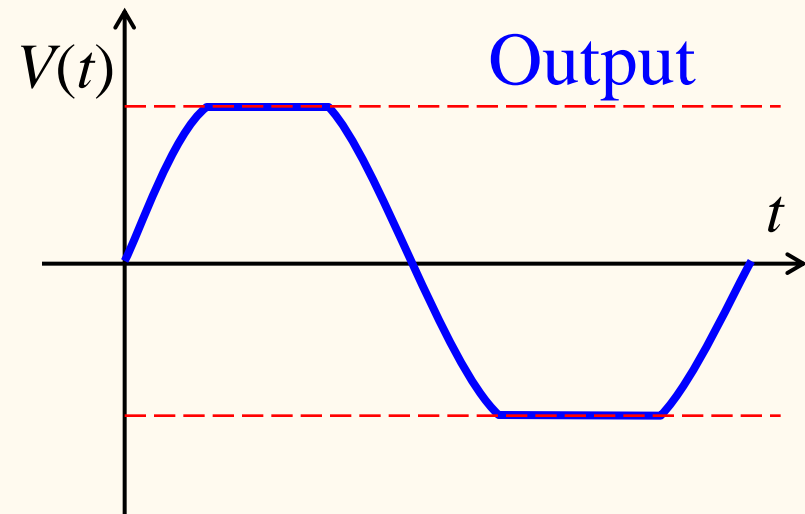
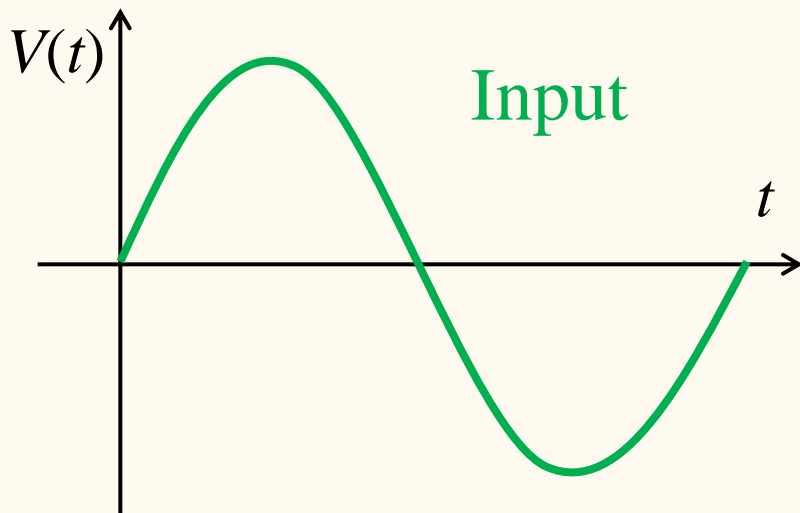
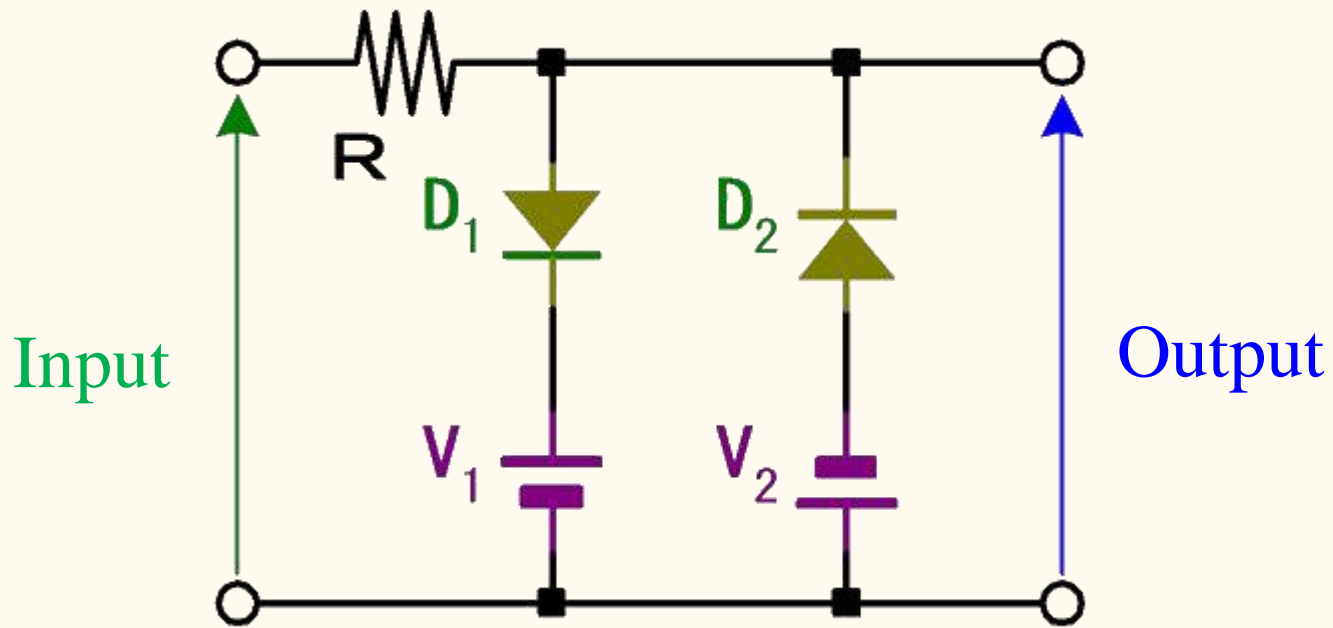
$$\begin{aligned} r(t) &= \underbrace{A_r \cos[\omega_c t + \phi(t)]}_{\text{Signal}} + \underbrace{n_l(t) \cos \omega_c t}_{\text{In phase}} - \underbrace{n_r(t) \sin \omega_c t}_{\text{Out of phase}} \\ &= A_r \cos[\omega_c t + \phi(t)] + A_n(t) \cos[\omega_c t + \phi_n(t)] \\ &= V_r(t) \cos[\omega_c t + \theta(t)] \quad (\theta(t) = \phi(t) + \underbrace{\phi_{\text{no}}(t)}_{\text{Phase noise}}) \end{aligned}$$

$$V_r(t) = \sqrt{A_r^2 + A_n^2(t) + 2A_r A_n(t) \cos[\phi_n(t) - \phi(t)]},$$

$$\phi_{\text{no}}(t) = \arctan \frac{A_n(t) \sin[\phi_n(t) - \phi(t)]}{A_r + A_n(t) \cos[\phi_n(t) - \phi(t)]}$$

Time-dependent part in  $V_r(t)$  can be cut with a limiter circuit.

## 6.3.5 Modulation and noise (Diode limiter)



## 6.3.5 Modulation and noise

$$A_r \gg A_n(t)$$

$$\phi_{\text{no}} \cong \arctan \left[ \frac{A_n(t)}{A_r} \sin[\phi_n(t) - \phi(t)] \right] \cong \frac{A_n(t)}{A_r} \sin[\phi_n(t) - \phi(t)]$$

Noise: White, Power spectrum density  $n_a/2$ , Band width  $\omega_B$

$$\text{Noise power: } N_i = \frac{n_a \omega_B}{2\pi} \quad \text{Signal power: } \frac{A_r^2}{2} \quad \frac{S_i}{N_i} = \frac{\pi A_r^2}{n_a \omega_B}$$

---

Phase modulation

$$\phi[t, f(t)] = k_p f(t)$$

Averaged signal power:  $k_p^2 \langle f^2 \rangle$

Averaged noise power:  $N_{\text{oPM}} \cong \frac{1}{A_r^2} \langle A_n(t)^2 \sin^2[\phi_n(t) - \phi(t)] \rangle$

$\phi_n(t)$ : Uniform in  $[0, 2\pi]$   $\rightarrow$  ignored

$$N_{\text{oPM}} \cong \frac{1}{A_r^2} \langle A_n(t)^2 \sin^2 \phi(t) \rangle = \frac{n_a \omega_w}{\pi A_r^2}$$

## 6.3.5 Modulation and noise

$$f(t) = A_p \cos \omega_p t, \quad \beta \equiv k_p A_p \rightarrow S_o = \frac{\beta^2}{2}, \quad \omega_B = 2(\beta + \xi)\omega_w \quad (1 \leq \xi \leq 2)$$

$$\frac{S_o}{N_o} = \frac{\beta^2}{2} \frac{\pi A_r^2}{n_a \omega_w} = \frac{\beta^2}{2} \frac{\omega_B}{\omega_w} \frac{\pi A_r^2}{n_a \omega_B} = \beta^2 (\beta + \xi) \frac{S_i}{N_i}$$

### Frequency modulation

Demodulated output  $d\theta/dt$       Signal, power:  $k_f f(t)$ ,  $k_f^2 \langle f^2 \rangle$

$$N_{o\text{FM}} = \left\langle \frac{dn_{no}}{dt} \right\rangle = \frac{1}{A_r^2} \left\langle \frac{dn_l}{dt} \right\rangle = \frac{1}{A_r^2} \int_{-\omega_w}^{\omega_w} n_a \omega^2 \frac{d\omega}{2\pi} = \frac{n_a \omega_w^3}{3\pi A_r^2}$$

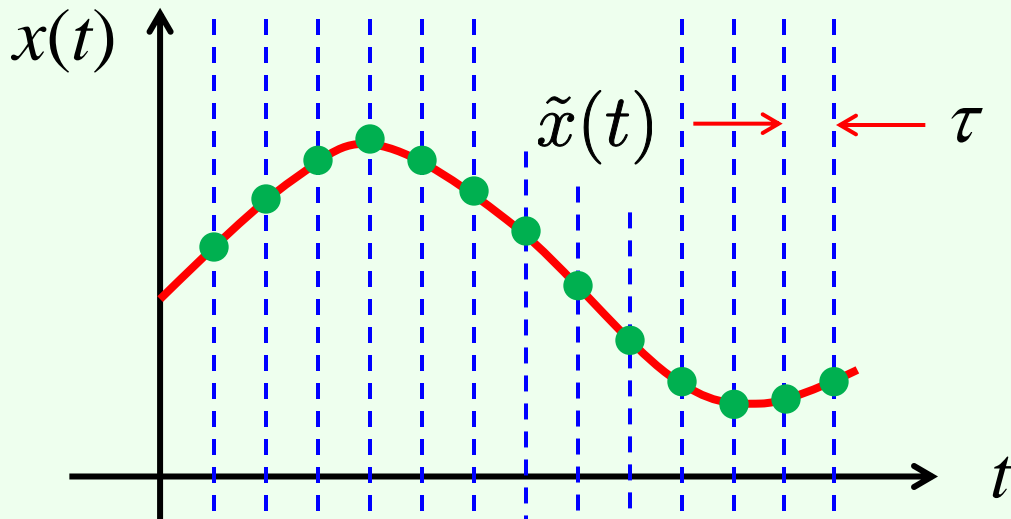
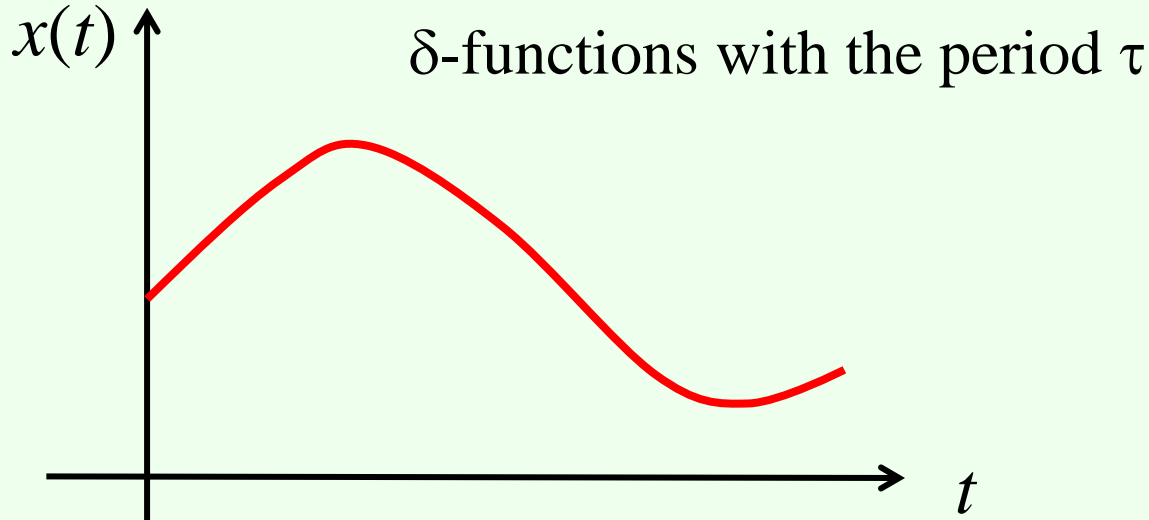
$$\beta \equiv k_f A_p / \omega_w \quad \frac{S_o}{N_o} = 3\beta^2 (\beta + \xi) \frac{S_i}{N_i}$$

$$\left. \frac{S_o}{N_o} \right|_{\text{FM}} = 3\beta^2 \left. \frac{S_o}{N_o} \right|_{\text{AM}}, \quad \left. \frac{S_o}{N_o} \right|_{\text{PM}} = \beta^2 \left. \frac{S_o}{N_o} \right|_{\text{AM}}$$

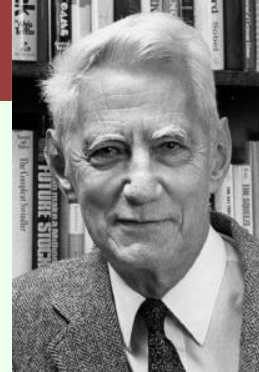
# 6.4 Discrete signal

## 6.4.1 Sampling theorem

Sampled signal  $\tilde{x}(t) = x(t)\delta_\tau(t)$



Isao Someya  
1915-2007

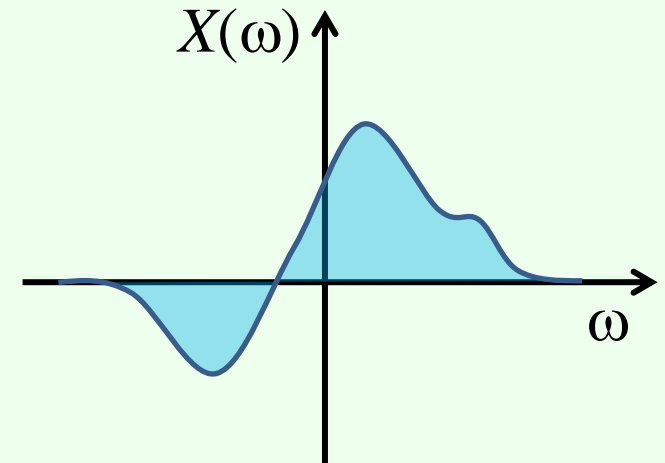


Claude Shannon  
1916-2001

1928 H. Nyquist

1949 C. Shannon

染谷勲



## 6.4.1 Sampling theorem

$$\begin{aligned}\delta_\tau(t) &= \sum_{j=-\infty}^{\infty} \delta(t - j\tau) = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{\tau} \int_{-\pi/\tau}^{\pi/\tau} \delta(s) ds \right] \exp\left(-in\frac{2\pi}{\tau}t\right) \\ &= \frac{1}{\tau} \sum_{n=-\infty}^{\infty} \exp\left(-in\frac{2\pi}{\tau}t\right)\end{aligned}$$

$$\begin{aligned}\mathcal{F}\{\delta_\tau(t)\} &= \int_{-\infty}^{\infty} \left[ \frac{1}{\tau} \sum_{n=-\infty}^{\infty} e^{-in(2\pi/\tau)t} \right] e^{i\omega t} dt = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[i\left(\omega - n\frac{2\pi}{\tau}\right)t\right] dt \\ &= \frac{2\pi}{\tau} \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\frac{2\pi}{\tau}\right) = \frac{2\pi}{\tau} \delta_{2\pi/\tau}(\omega)\end{aligned}$$

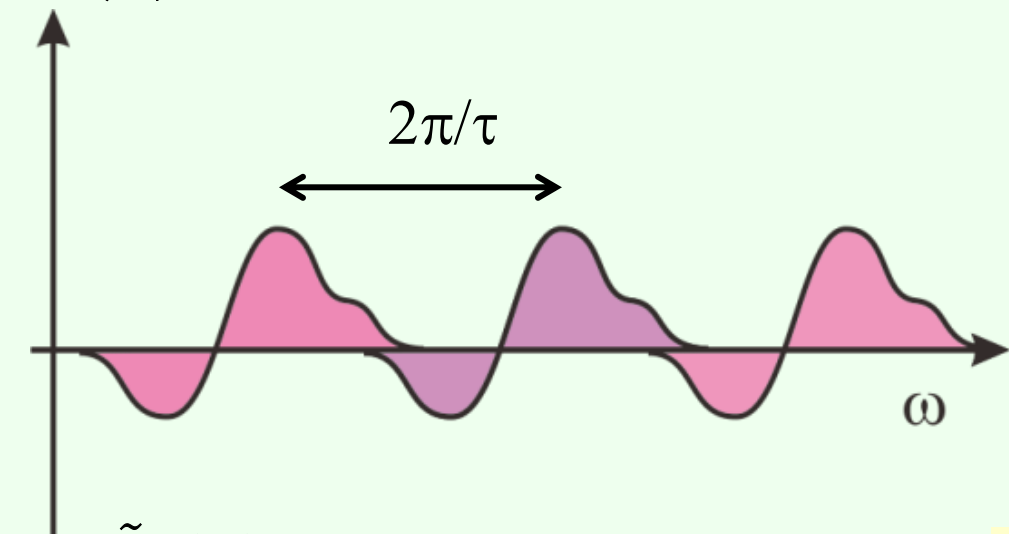
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$$\mathcal{F}\{x(t)\} = X(\omega), \quad \mathcal{F}\{\tilde{x}_\tau(t)\} = \tilde{X}_\tau(\omega)$$

$$\begin{aligned}\tilde{X}_\tau(\omega) &= \frac{1}{2\pi} X(\omega) * \frac{2\pi}{\tau} \delta_{2\pi/\tau}(\omega) = \frac{1}{\tau} X(\omega) * \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\frac{2\pi}{\tau}\right) \\ &= \frac{1}{\tau} \int_{-\infty}^{\infty} X(\omega') \left\{ \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\frac{2\pi}{\tau} - \omega'\right) \right\} d\omega' = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} X\left(\omega - n\frac{2\pi}{\tau}\right)\end{aligned}$$

# 6.4.1 Sampling theorem

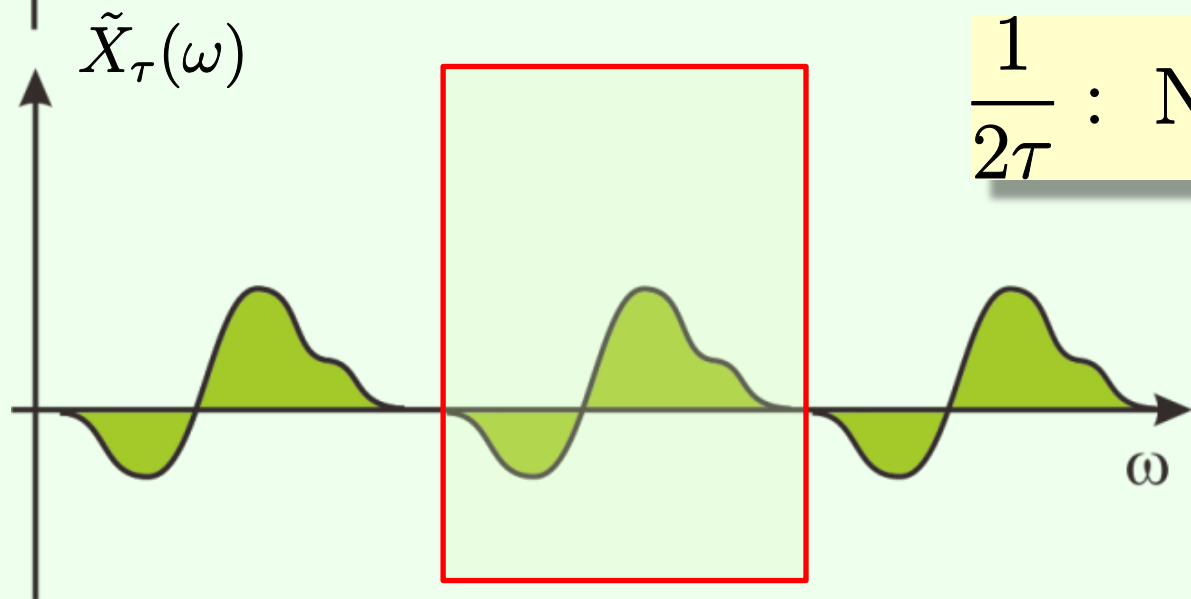
$\tilde{X}_\tau(\omega)$  “Cutting out” the frequency spectrum



$\omega_h$ : Highest frequency in  $\tilde{X}_\tau(\omega)$

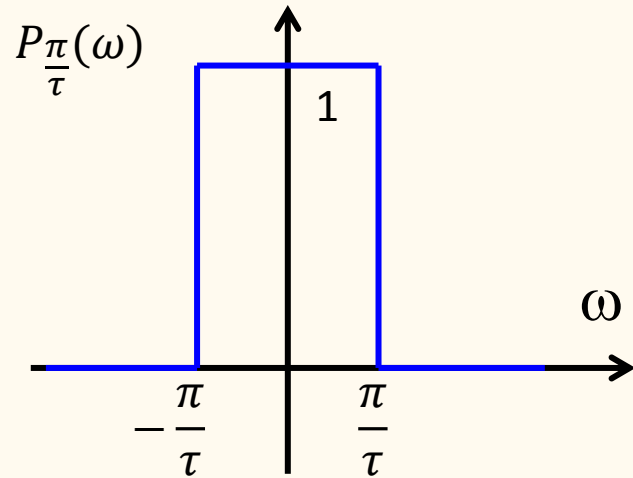
$$\frac{2\pi}{\tau} > 2\omega_h, \quad \tau < \frac{\pi}{\omega_h}$$

$\frac{1}{2\tau}$  : Nyquist frequency

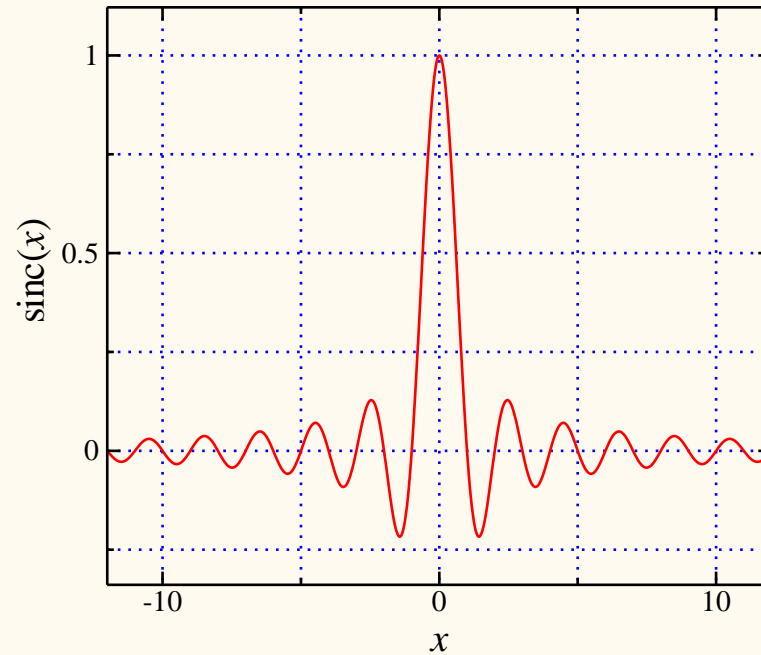


## 6.4.1 Sampling theorem: Reconstructing signal

$$P_{\pi/\tau}(\omega) = \begin{cases} 1 & |\omega| \leq \frac{\pi}{\tau}, \\ 0 & |\omega| > \frac{\pi}{\tau} \end{cases}$$



$$x(t) = \mathcal{F}^{-1}\{\tau P_{\pi/\tau}(\omega)\tilde{X}_\tau(\omega)\}$$

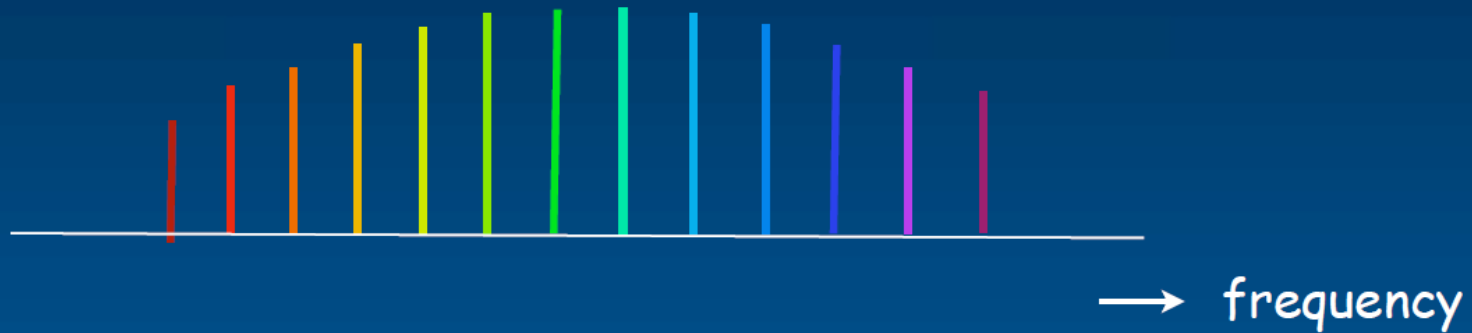
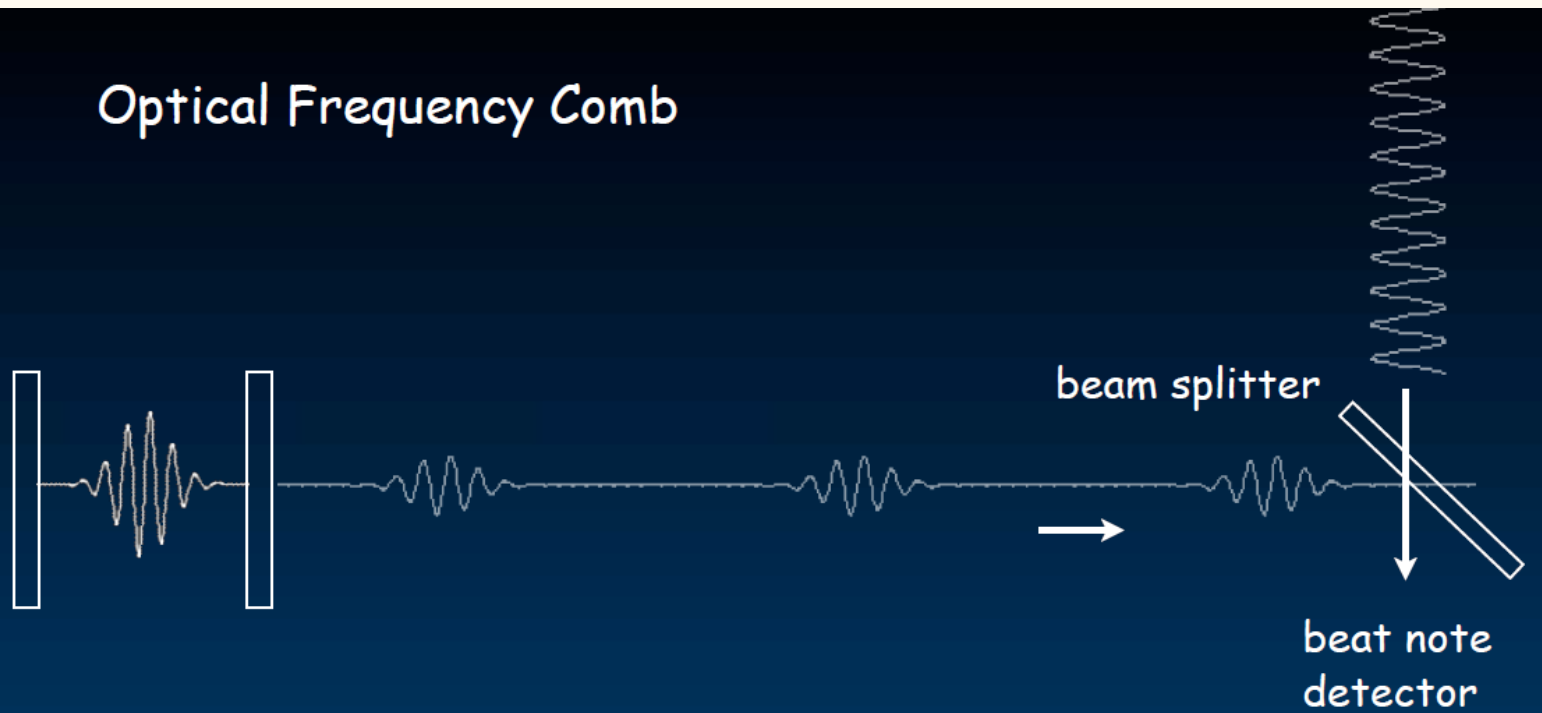


$$\begin{aligned} x(t) &= \tau \frac{1}{\tau} \text{sinc}\left(\frac{t}{\tau}\right) * \tilde{x}_\tau(t) = \text{sinc}\left(\frac{t}{\tau}\right) * \sum_{n=-\infty}^{\infty} x(t)\delta(t - n\tau) \\ &= \int_{-\infty}^{\infty} \text{sinc}\left(\frac{s}{\tau}\right) \sum_{n=-\infty}^{\infty} x(t-s)\delta(t - n\tau - s)ds = \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{t - n\tau}{\tau}\right) x(n\tau) \end{aligned}$$

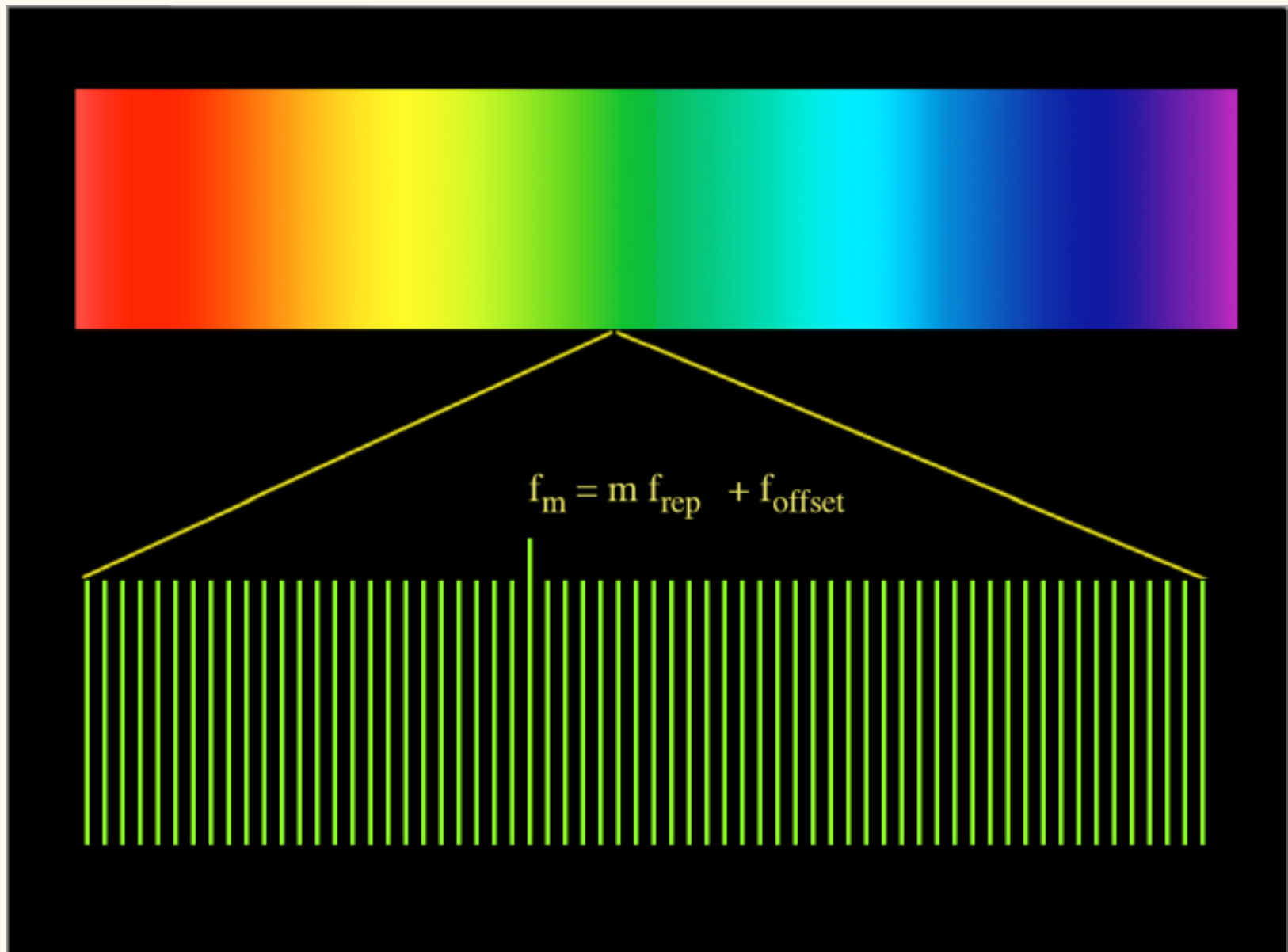


# Optical Frequency Comb

## Optical Frequency Comb



# Frequency Comb



# Measurement of the Doppler effect in cosmic expansion

