Electric Circuit for Physicists
電子回路論　第9回

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Review: Distributed constant circuit concept

1. In what case we need to consider distributed constant circuits?

Characteristic sizes of devices $\gtrsim$ wavelength of electromagnetic signal

2. A typical scheme to make the shift for distributed circuit

Lumped constant circuit → 1. Connection of unit circuits 2. Taking the infinitesimal limit → Distributed constant circuit

3. Distributed constant circuits : transmission lines

Coaxial cables, Lecher lines, micro-strip lines

Characteristic impedance

Transmission line: Series of two-terminal pair circuits

Width → 0, Number → ∞

$$dV = -JZdx, \quad dJ = -VYdx$$

Z : Impedance per unit length, Y : Admittance per unit length

$$\frac{d^2J}{dx^2} = YZJ, \quad \frac{d^2V}{dx^2} = YZV$$ Telegraphic equation

$$\kappa \equiv \sqrt{YZ} \qquad \begin{cases} J(x,t)_{\mp} = J_{\mp}(0,t)\exp(\pm\kappa x) \\ V_{\mp}(x,t) = V_{\mp}(0,t)\exp(\pm\kappa x) \end{cases}$$

−: Progressive, +: Retrograde

$$\frac{V_{\pm}}{J_{\pm}} = \mp\frac{Z}{\kappa} = \mp\sqrt{\frac{Z}{Y}} \equiv \mp Z_0$$ Characteristic impedance

Pure reactance transmission line

Pure reactance $Y = i\omega C,\ Z = i\omega L$

$$\kappa = \sqrt{-\omega^2 LC} = i\frac{\omega}{\omega_0}, \quad \omega_0 \equiv \frac{1}{\sqrt{LC}}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$ Characteristic impedance

For coaxial cable:

From Maxwell equation $$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\log\frac{b}{a}$$

cf. Characteristic impedance of vacuum

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 376\Omega$$

Connection and termination

At $x = 0$: $\begin{cases} J = J_+ + J_- \quad \text{(definition right positive)} \\ \text{progressive} \quad \text{retrograde} \\ \\ V = V_+ + V_- = Z_0(J_+ - J_-) \end{cases}$

$$Z_1 = \frac{V}{J} = \frac{J_+ - J_-}{J_+ + J_-} Z_0$$

Reflection coefficient: $r = \frac{V_-}{V_+} = -\frac{J_-}{J_+} = \frac{Z_1 - Z_0}{Z_1 + Z_0}$

$Z_1 = Z_0$: no reflection, i.e., impedance matching

$Z_1 = +\infty$ (open circuit end) : $r = 1$, i.e., free end

$Z_1 = 0$ (short circuit end) : $r = -1$, i.e., fixed end

Connection and termination

Finite reflection → Standing wave

Voltage-Standing Wave Ratio (VSWR): $= \dfrac{1+|r|}{1-|r|}$

At $x = -l$

$$\left.\begin{aligned} V &= V_{+0}\exp(\kappa l) + V_{-0}\exp(-\kappa l) = [J_{+0}\exp(\kappa l) - J_{-0}\exp(-\kappa l)]Z_0 \\ J &= J_{+0}\exp(\kappa l) + J_{-0}\exp(-\kappa l) \end{aligned}\right\}$$

$$Z_l = \frac{V}{J} = \frac{J_{+0}e^{\kappa l} - J_{-0}e^{-\kappa l}}{J_{+0}e^{\kappa l} + J_{-0}e^{-\kappa l}} Z_0$$

Reflection coefficient:

$$r_l = \frac{V_-}{V_+} = \frac{V_{-0}e^{-\kappa l}}{V_{+0}e^{\kappa l}} = r\exp(-2\kappa l)$$

Exercise 4-3

l =1km の伝送線路がある．終端側を短絡したところ，電源側から測定したインピダンスは0.6i Ωであった．一方，終端側を開放して電源側からアドミタンスを測定すると4x10^{-6}i Sであった．

この伝送線路の特性インピダンスを求めよ．

Consider a transmission line with the length l = 1km. First we short-circuited the end and measured the impedance from the signal source and obtained 0.6i Ω. Next we opened the end and measured the admittance from the signal source and obtained 4x10^{-6}i S.

What is the characteristic impedance of the transmission line?

Connection and termination

Transmission line connection.
Characteristic impedance Z_0, Z_0'

At the connection point, only the local relation between V and J affects the reflection coefficient.

The local impedance from the left hand side is Z_0'.

$$r = \frac{Z_0' - Z_0}{Z_0' + Z_0}$$

5.2.3 Smith chart, Immittance chart

End impedance Z_1: Normalized end impedance $Z_n \equiv Z_1/Z_0$

$$Z_n = x + iy, \quad r = u + iw \quad (x, y, u, w \in \mathbb{R})$$

$$u + iw = r = \frac{Z_n - 1}{Z_n + 1} = \frac{(x-1) + iy}{(x+1) + iy}$$

$$\left. \begin{aligned} x - 1 &= (x+1)u - yw \\ y &= yu + w(x+1) \end{aligned} \right\}$$

x: constant → $$\left(u - \frac{x}{x+1}\right)^2 + w^2 = \frac{1}{(x+1)^2}$$ constant resistance circle

y: constant → $$(u-1)^2 + \left(w - \frac{1}{y}\right)^2 = \frac{1}{y^2}$$ constant reactance circle

5.3 S matrix (S parameters)

How to treat multipoint (crossing point) systematically?

Transmission lines: wave propagating modes → Channels

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} S_{11} & \cdots & S_{1i} & \cdots & S_{1n} \\ \vdots & \ddots & & & \vdots \\ S_{i1} & & S_{ii} & & S_{in} \\ \vdots & & & \ddots & \vdots \\ S_{n1} & \cdots & S_{ni} & \cdots & S_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} = \mathrm{S}\boldsymbol{a}$$

S → S matrix

Reciprocity $$S_{ij} = S_{ji}$$

Unitarity $$\sum_j S_{ji} S_{jk}^* = \delta_{ik}$$

In case, no dissipation, no amplification.

Two-terminal pair S matrix $$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

5.3 S-matrix (S-parameters)

Application to electric transmission lines

What is conserved in the propagation with no dissipation? Energy

$$\begin{cases} a_n = \dfrac{V_{n+}}{\sqrt{Z_{0n}}} = J_{n+}\sqrt{Z_{0n}}, \\ b_n = \dfrac{V_{n-}}{\sqrt{Z_{0n}}} = J_{n-}\sqrt{Z_{0n}} \end{cases}$$

Incident power wave

Reflected (transmitted) power wave

$$|a_n|^2 = \frac{|V_{n+}|^2}{Z_{0n}} = |J_{n+}|^2 Z_{0n}$$

(assumption: common wave velocity
$\rightarrow Z_0$ common for channels)

Simplest case: series impedance Z_S

Terminate 2 with $Z_0 \rightarrow a_2 = 0$

$$S_{11} = \frac{V_{1-}}{V_{1+}} = \frac{V_1 - Z_0 J_1}{V_1 + Z_0 J_1} = \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{(Z_S + Z_0) - Z_0}{(Z_S + Z_0) + Z_0} = \frac{Z_S}{Z_S + 2Z_0}$$

$$S_{21} = \frac{V_{2-}}{V_{1+}} = \frac{V_2}{E_1/2} = \frac{2V_2}{V_1 + Z_0 J_1} = \frac{2Z_0 J_1}{(Z_S + Z_0)J_1 + Z_0 J_1} = \frac{2Z_0}{Z_S + 2Z_0}$$

5.3 S-matrix (S-parameters)

Terminate 1 with $Z_0 \rightarrow a_1 = 0$ (should be symmetric)

$$S_{12} = \frac{2V_1}{V_2 + Z_0 J_2} = \frac{2Z_0 J_2}{(Z_S + Z_0)J_2 + Z_0 J_2} = \frac{2Z_0}{Z_S + 2Z_0}$$

$$S_{22} = \frac{Z_S}{Z_S + 2Z_0}$$

Generally

$$\mathrm{S} = \frac{1}{\det Z} \times \begin{pmatrix} (Z_{11} - Z_0)(Z_{22} + Z_0) - Z_{12}Z_{21} & 2Z_0 Z_{12} \\ 2Z_0 Z_{21} & (Z_{11} + Z_0)(Z_{22} - Z_0) - Z_{12}Z_{21} \end{pmatrix}$$

Cascade connection of S-matrices

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{S} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} r_{\mathrm{L}} & t_{\mathrm{R}} \\ t_{\mathrm{L}} & r_{\mathrm{R}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$r_{L,R}$, $t_{L,R}$: complex reflection, transmission coefficients satisfying

$$T_{\mathrm{L,R}} = |t_{\mathrm{L,r}}|^2 = 1 - R_{\mathrm{L,R}} = 1 - |r_{\mathrm{L,R}}|^2$$

$$\mathbf{S}_{\mathrm{AB}} = \begin{pmatrix} r_{\mathrm{L}}^{\mathrm{AB}} & t_{\mathrm{R}}^{\mathrm{AB}} \\ t_{\mathrm{L}}^{\mathrm{AB}} & r_{\mathrm{R}}^{\mathrm{AB}} \end{pmatrix}$$

$$= \begin{pmatrix} r_{\mathrm{L}}^{\mathrm{A}} + t_{\mathrm{R}}^{\mathrm{A}} r_{\mathrm{L}}^{\mathrm{B}} \left(I - r_{\mathrm{R}}^{\mathrm{A}} r_{\mathrm{L}}^{\mathrm{B}}\right)^{-1} t_{\mathrm{L}}^{\mathrm{A}} & t_{\mathrm{R}}^{\mathrm{A}} \left(I - r_{\mathrm{L}}^{\mathrm{B}} r_{\mathrm{R}}^{\mathrm{A}}\right)^{-1} t_{\mathrm{R}}^{\mathrm{B}} \\ t_{\mathrm{L}}^{\mathrm{B}} \left(I - r_{\mathrm{R}}^{\mathrm{A}} r_{\mathrm{L}}^{\mathrm{B}}\right)^{-1} t_{\mathrm{L}}^{\mathrm{A}} & r_{\mathrm{R}}^{\mathrm{B}} + t_{\mathrm{L}}^{\mathrm{B}} \left(I - r_{\mathrm{R}}^{\mathrm{A}} r_{\mathrm{L}}^{\mathrm{B}}\right)^{-1} r_{\mathrm{R}}^{\mathrm{A}} t_{\mathrm{R}}^{\mathrm{B}} \end{pmatrix}$$

$$(I - r_R^A r_L^B)^{-1} = I + r_R^A r_L^B + (r_R^A r_L^B)^2 + \cdots$$

Conduction channels in quantum transport

Rolf Landauer

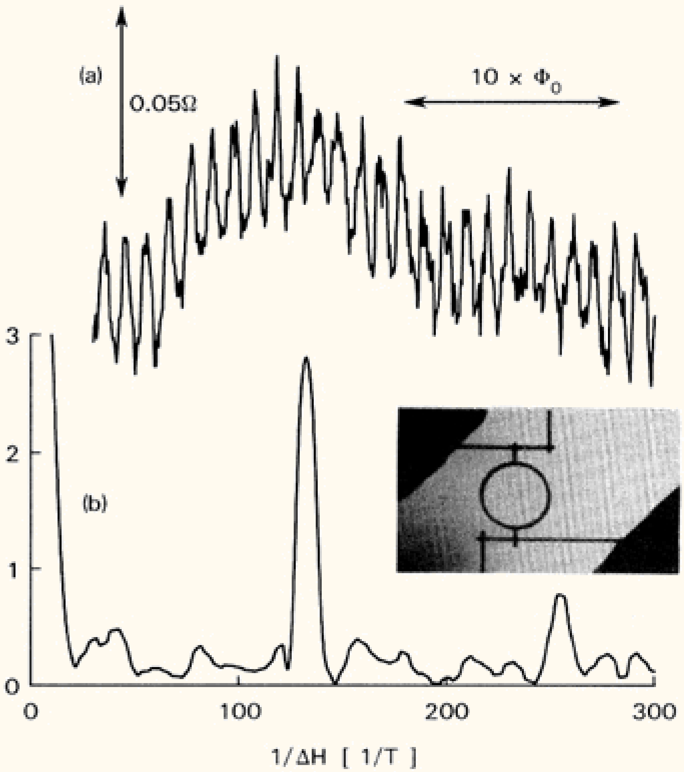
Electron (quantum mechanical) waves also have propagating modes in solids.

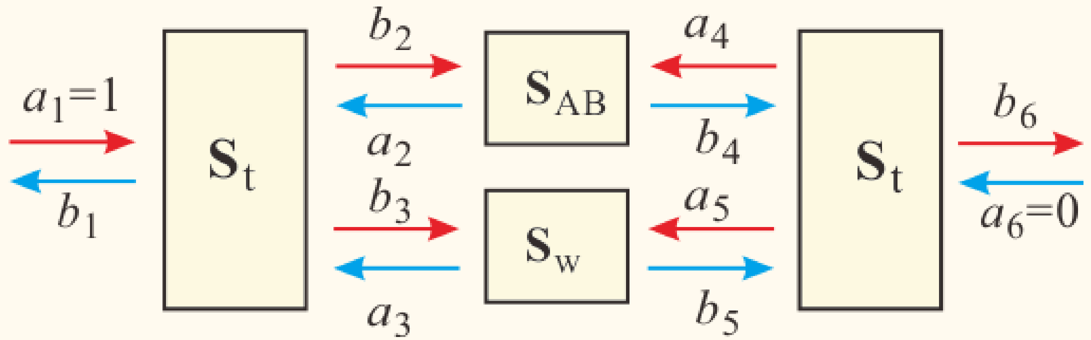
→ Conduction channel

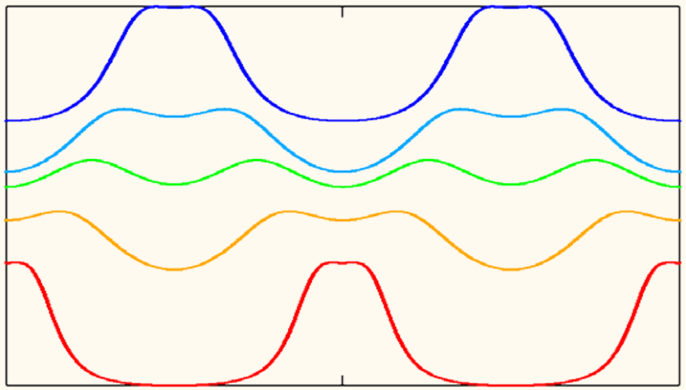
Landauer eq.:

the conductance of a single perfect quantum channel is $\dfrac{e^2}{h}$

AB ring S-matrix model

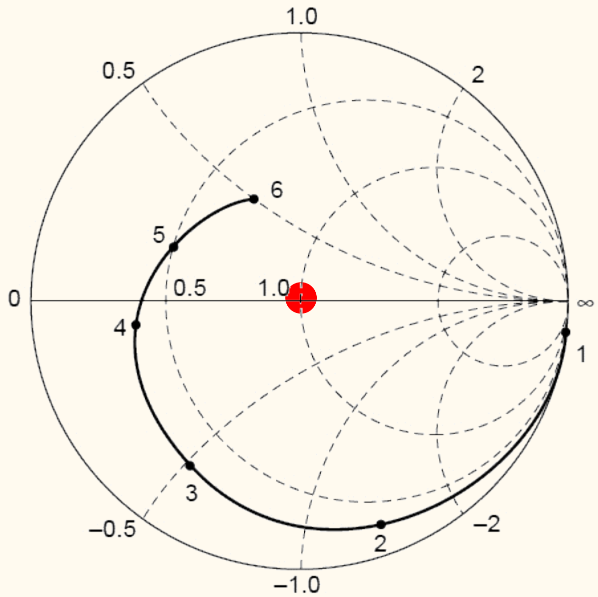






S-parameter representation of high-frequency devices

Ex) NE76084 MES FET 0.5〜18GHz



S_{11}

Needs impedance matching circuits

If we know Z-parameters:

From Ho-Thevenin theorem

$$Z_{\mathrm{S}} = Z^{\mathrm{i}}_{22} - \frac{Z^{\mathrm{i}}_{12}Z^{\mathrm{i}}_{21}}{50 + Z^{\mathrm{i}}_{11}}, \quad Z_{\mathrm{L}} = Z^{\mathrm{o}}_{11} - \frac{Z^{\mathrm{o}}_{12}Z^{\mathrm{o}}_{21}}{50 + Z^{\mathrm{o}}_{22}}$$

$\{Z_{ij}\}$: BJT (FET) Z-parameters, again Ho-Thevenin says

$$Z_l = Z_{11} - \frac{Z_{12}Z_{21}}{Z_{\mathrm{L}} + Z_{22}}, \quad Z_r = Z_{22} - \frac{Z_{12}Z_{21}}{Z_{\mathrm{S}} + Z_{11}}$$

Impedance matching with S-parameters

Generally the unitarity does not hold for amplification.

$$R_{\mathrm{in}} = S_{11} + \frac{S_{12}S_{21}R_{\mathrm{L}}}{1 - S_{22}R_{\mathrm{L}}} \qquad R_{\mathrm{out}} = S_{22} + \frac{S_{12}S_{21}R_{\mathrm{S}}}{1 - S_{11}R_{\mathrm{S}}}$$

Matching condition: $R_{\mathrm{L}} = R^*_{\mathrm{out}}, \quad R_{\mathrm{S}} = R^*_{\mathrm{in}}$

Solution $$R_{\mathrm{S}} = \frac{B_1 \pm \sqrt{B_1^2 - 4|M|^2}}{2M}, \quad R_{\mathrm{L}} = \frac{B_2 \pm \sqrt{B_2^2 - 4|N|^2}}{2N}$$ with

$$B_1 = 1 + |S_{11}|^2 - |S_{22}|^2 - |\mathrm{det}S|^2, \quad B_2 = 1 - |S_{11}|^2 + |S_{22}|^2 - |\mathrm{det}S|^2,$$
$$N = S_{22} - S^*_{11}\mathrm{det}S, \quad M = S_{11} - S^*_{22}\mathrm{det}S$$

maximum available power gain $$G_{\mathrm{max}} = \left|\frac{S_{21}}{S_{12}}\right| (K - \sqrt{K^2 - 1})$$

$$K = \frac{1 + |\mathrm{det}S|^2 - |S_{11}|^2 - |S_{22}|^2}{2|S_{12}S_{21}|}$$ stability factor

Impedance matching designer

http://home.sandiego.edu/~ekim/e194rfs01/jwmatcher/matcher2.html

http://leleivre.com/rf_lcmatch.html

5.4 Giving mass to LC transmission line

Constant finite mass: $E = \hbar\omega \propto k^2$

(Schrodinger eq.: Parabolic partial differential equation)

Coupling between linear dispersions: mass mechanism *cf*. Higgs

C: capacitance per unit length
L: inductance per inverse unit length
K: inductance per unit length

$\frac{1}{\sqrt{LC}} = \omega_0$ unchanged with $dx \to 0$

$$Z = i\omega K, \quad Y = \frac{1-(\omega/\omega_0)^2}{i\omega L}$$

$$ik = \kappa = \sqrt{YZ} = i\sqrt{\frac{K}{L}\left[\left(\frac{\omega}{\omega_0}\right)^2 - 1\right]}$$

$$\eta^2 \equiv \frac{K}{L}$$

5.4 Giving mass to LC transmission line

$$\omega \gg \omega_0 \to k \sim \eta \frac{\omega}{\omega_0}$$ No dispersion

Velocity: $$c^* = \frac{\omega}{k} = \frac{\omega_0}{\eta} = \frac{1}{\sqrt{KC}}$$

$$\omega \sim \omega_0 \quad \omega = \omega_0 + \delta\omega$$

$$k^2 \approx 2\eta^2 \frac{\delta\omega}{\omega_0} \qquad \therefore \epsilon \equiv \hbar\delta\omega = \frac{\hbar k^2}{2(\eta^2/\omega_0)} = \frac{\hbar^2 k^2}{2m^*}$$

$$m^* \equiv \frac{\hbar\eta^2}{\omega_0}$$

$$E_0 = \hbar\omega_0 = \frac{\hbar\eta^2}{\omega_0} \cdot \left(\frac{\omega_0}{\eta}\right)^2 = m^* c^{*2}$$

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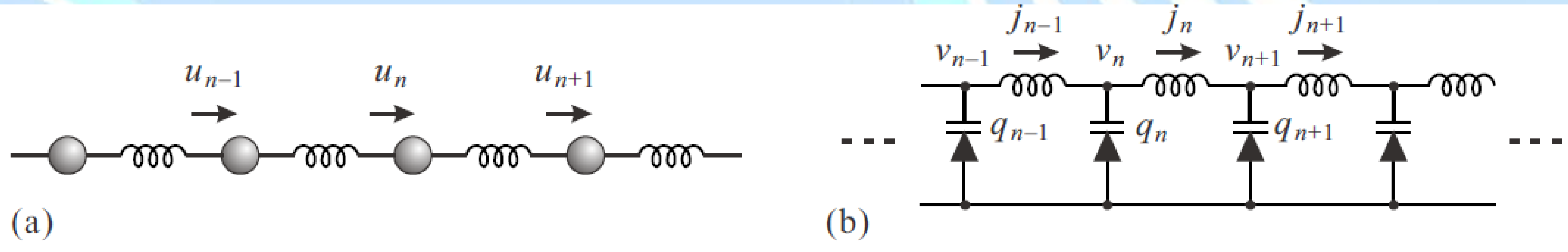
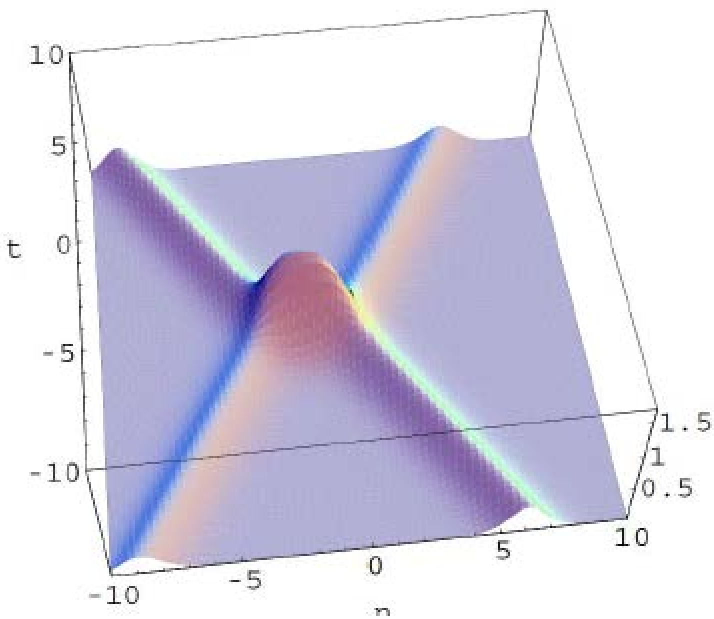
Toda lattice





$$u_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} - 1,$$

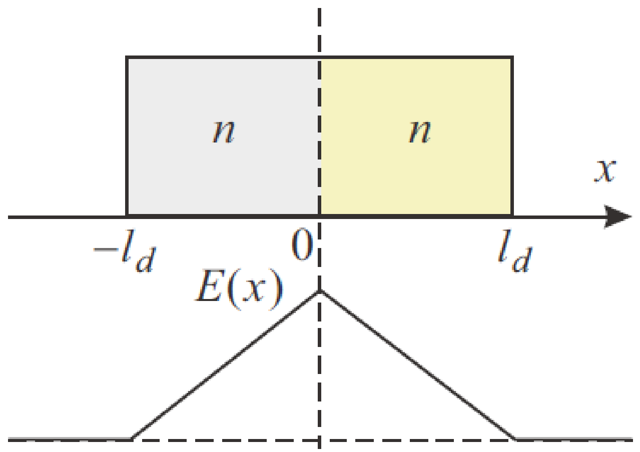
$$\tau_n = 1 + e^{2\eta_1} + e^{2\eta_2} + A_{12}e^{2(\eta_1+\eta_2)},$$

$$\eta_i = \kappa_i n + \sigma_i\omega_i t + \delta_i, \quad \sigma_i = \pm 1, \ \ \omega_i = \sinh\kappa_i,$$

$$A_{12} = \frac{ab\sinh^2(\kappa_1-\kappa_2) - m(\sigma_1\omega_1-\sigma_2\omega_2)^2}{m(\sigma_1\omega_1+\sigma_2\omega_2)^2 - ab\sinh^2(\kappa_1+\kappa_2)}$$

Vari-cap

Varicap BB505

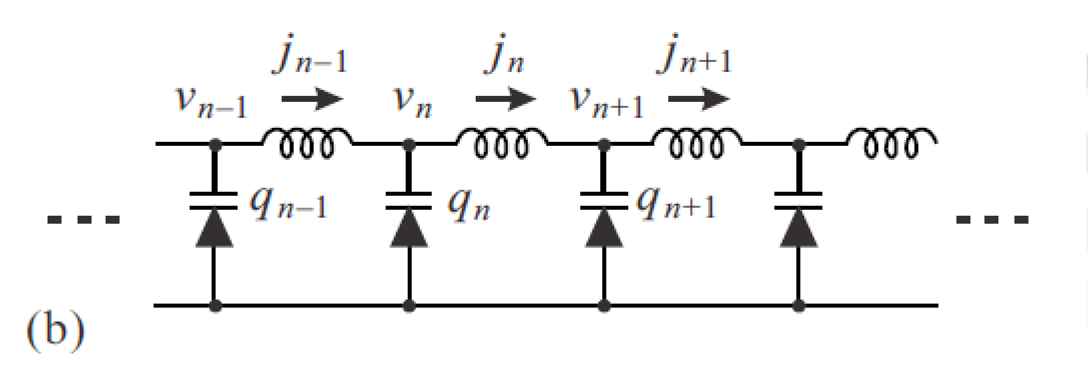


$$V_\mathrm{b} = \frac{en}{\epsilon}\int_{-l_d}^{0} 2(x+l_d)dx + \frac{en}{\epsilon}\int_{0}^{l_d} 2(l_d - x)dx = \frac{2enl_d^2}{\epsilon}$$

$$V + V_\mathrm{b} = \frac{2en}{\epsilon}\left(l_d + \frac{Q}{nS}\right)^2 \quad \therefore C = \frac{dQ}{dV} = \sqrt{\frac{\epsilon}{2en}}\frac{nS}{\sqrt{V+V_\mathrm{b}}}$$

$$V + V_\mathrm{b} = V_0 + \delta V \qquad \delta V \to V$$

L-Varicap transmission line



$$L\frac{dJ_n}{dt} = v_n - v_{n-1},$$

$$\frac{dq_n}{dt} = J_{n-1} - J_n,$$

$$q_n = \int_0^{v_n} C(V)dV, \quad C(V) = \frac{Q(V_0)}{F(V_0) + V - V_0}$$

$$q_n = Q(V_0)\log\left[1 + \frac{V_n}{F(V_0)}\right] + \text{const.}$$

$$\frac{d^2}{dt^2}\log\left[1 + \frac{V_n}{F(V_0)}\right] = \frac{1}{LQ(V_0)}(V_{n-1} + V_{n+1} - 2V_n)$$

Solitons in non-linear circuit

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CIRCUIT IMPLEMENTATIONS OF SOLITON SYSTEMS*

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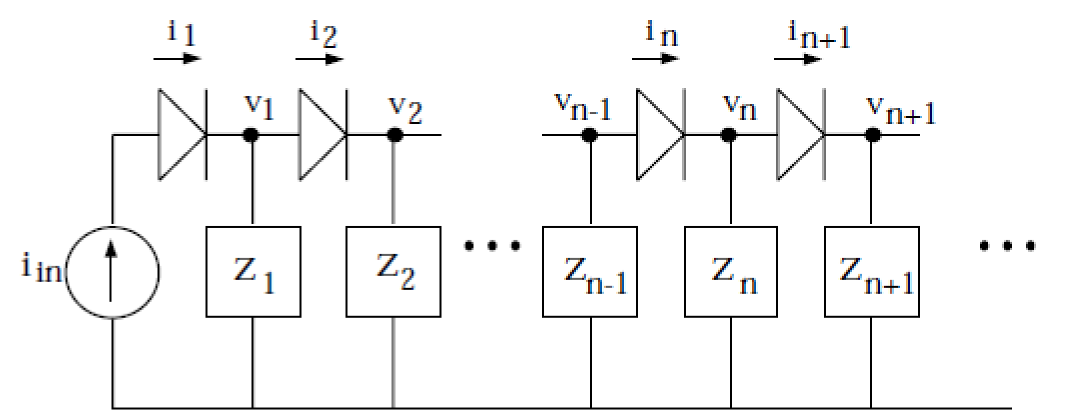
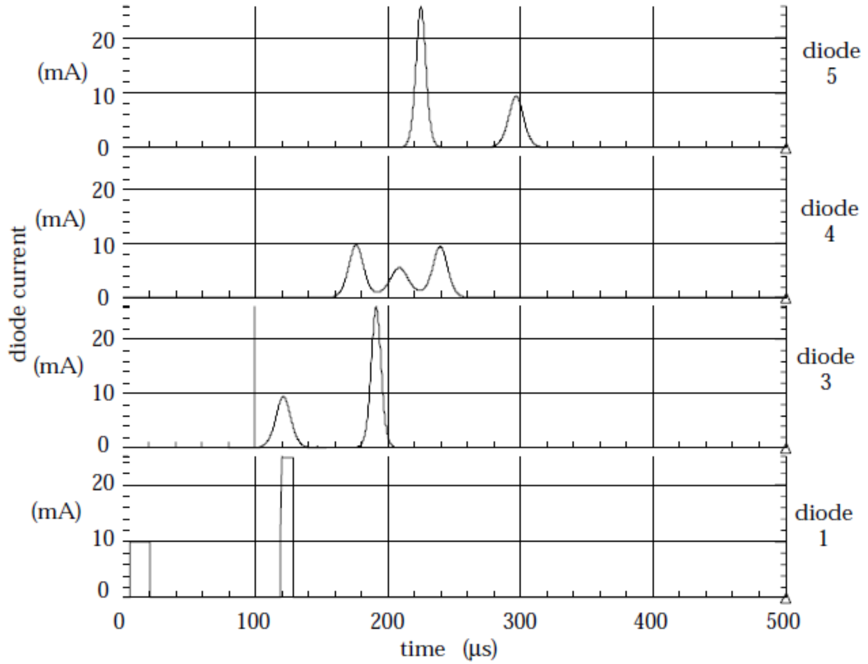
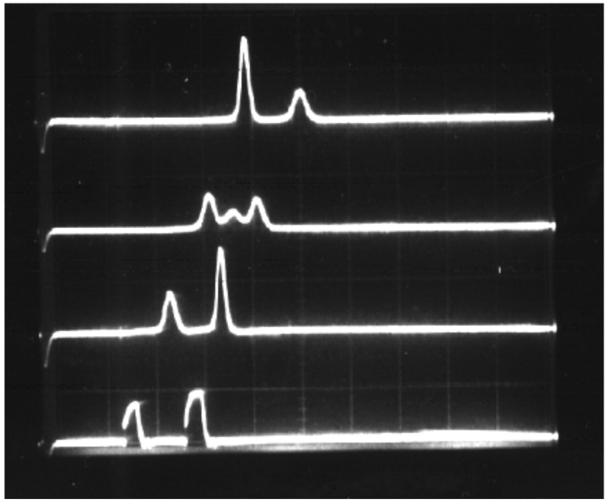






Toda lattice circuit, Soliton circuit

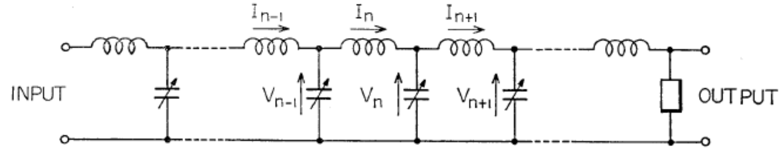
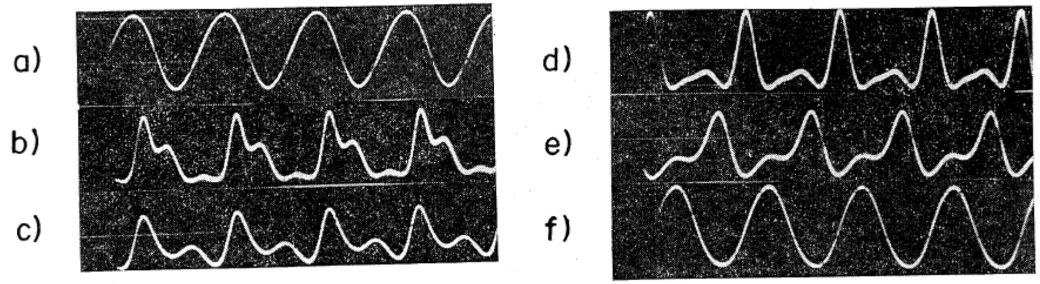


Fig. 1. A nonlinear network equivalent to a one-dimensional anharmonic lattice. The circuit element have an inductance $L=22\ \mu$H or capacitance $C(V)=27\ V^{-0.48}$ pF.



J. Phys. Soc. Japan **28** (1970) 1366~1367

Studies on Lattice Solitons by Using Electrical Networks

Ryogo Hirota and Kimio Suzuki

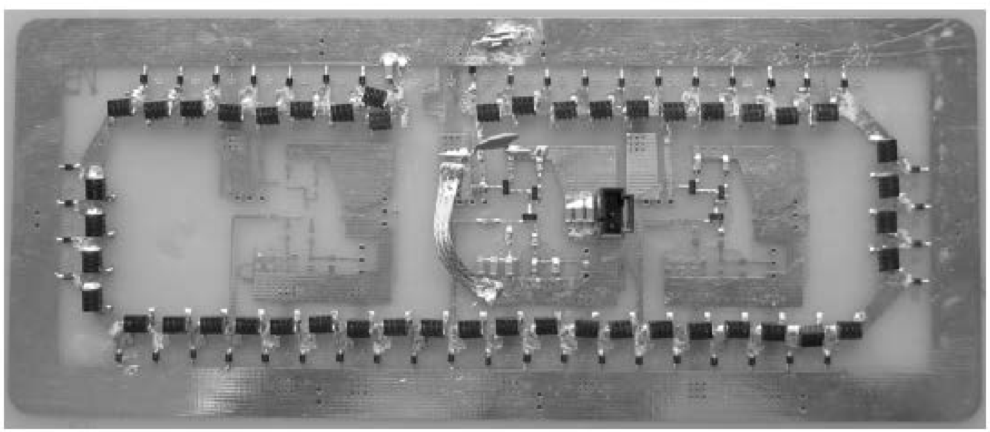


Fig. 16. Microwave soliton oscillator prototype.