Electric circuits for physicists (Appendix 1)

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1 A circuit model for amplifier with noise

Every amplificaition circuit *i.e.*, has noise sources in it. A way to take these noise sources into account is to assume noise power sources at the input of the amplifier as illustrated in Fig.1. Here the amplifier is assumed to have no noise. G_p being the power gain of the amplifier, then

$$\delta e_{\mathrm{intotal}}^2 = e_{\mathrm{out}}^2/G_\mathrm{p} = j_n^2 R^2 + e_\mathrm{R}^2 + e_a^2. \tag{1}$$

In such a model for amplifier with noise sources, the signal to noise ratio (**S/N ratio**) becomes worse by the amplification. Here the **noise figure** (**NF**) is defined as

$$\mathrm{NF} = 10\log_{10}\frac{(S/N)_\mathrm{in}}{(S/N)_\mathrm{out}} = 10\log_{10}\frac{S_\mathrm{in}N_\mathrm{out}}{S_\mathrm{out}N_\mathrm{in}}, \tag{2}$$

and expressed the decrease in the S/N ratio through the amplifier. Because

$$N_\mathrm{out} = G_\mathrm{p}\overline{e_\mathrm{N}^2},$$

the NF ratio is obtained as

$$\mathrm{NF} = 10\log_{10}\frac{S_\mathrm{in}G_\mathrm{p}\overline{e_\mathrm{N}^2}}{S_\mathrm{in}G_\mathrm{p}\overline{e_\mathrm{R}^2}} = 10\log_{10}\frac{\overline{e_\mathrm{N}^2}}{\overline{e_\mathrm{R}^2}} = 10\log_{10}\frac{\overline{e_n^2}+\overline{e_\mathrm{R}^2}+\overline{j_n^2}R^2}{\overline{e_\mathrm{R}^2}}. \tag{3}$$

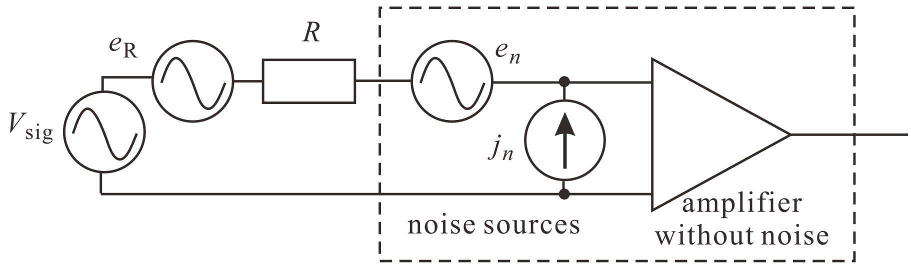
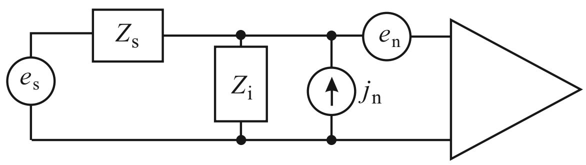


Figure 1: A circuit model of amplifier with noise sources

2 Concept of "noise matching"

In a previous lecture (21st, Nov.), we had a look on "power (impedance) matching" in an FET amplifier. Under the power matching condition, the output power gain is maximized by reducing the reactive part. However, as can be seen in Fig.1, this circuit also amplifies the noise and the power matching condition does not give the optimum S/N ratio. In the following, let me show a way to look for the maximum S/N condition.



In the left figure, Z_i is the input impedance of the amplifier. We write the noise temperature T_a at the input of amplifier and the resistance that matches to the noise power sources R_bs are

$$T_\mathrm{a} = \frac{\sqrt{\overline{e_n^2}\ \overline{j_n^2}}}{2k_\mathrm{B}}, \quad R_\mathrm{bs} = \sqrt{\frac{\overline{e_n^2}}{\overline{j_n^2}}}. \tag{4}$$

The noise temperature of the whole (left-hand-side circuit at the input of amplifier) $T_{\rm n}$ is then

$$T_{\rm n}=\left(1+\frac{{\rm Re}(1/Z_{\rm i})}{{\rm Re}(1/Z_{\rm s})}\right)\frac{T_{\rm a}}{2{\rm Re}Z}\left(\frac{|Z|^2}{R_{\rm bs}}+R_{\rm bs}\right),\quad \frac{1}{Z}\equiv\frac{1}{Z_{\rm i}}+\frac{1}{Z_{\rm s}}. \tag{5}$$

The condition to minimize $T_{\rm n}$ is given as

$$Z_{\rm i}=\frac{1}{R_{\rm bs}^{-1}-Z_{\rm s}^{-1}}, \tag{6}$$

and the minimun $T_{\rm n}$ is

$$T_{\rm n}=\left(1+\frac{{\rm Re}(1/Z_{\rm i})}{{\rm Re}(1/Z_{\rm s})}\right)T_{\rm a}. \tag{7}$$

Equation (6) is a **noise matching** condition.

Note that this is not almighty method. Rather, it is saying that we need to select the characteristics of amplifier according to the signal source impedance.

3 Noise figure circle in Smith chart

An easier tuning method is to minimize the nose figure of the amplifier. We shoud, again be careful about the noise figure. Our ultimate purpose is to obtain the best S/N ratio and not to obtain the minimum NF[2]. However this is not always easy and easier "minimum NF" is also frequently adopted.

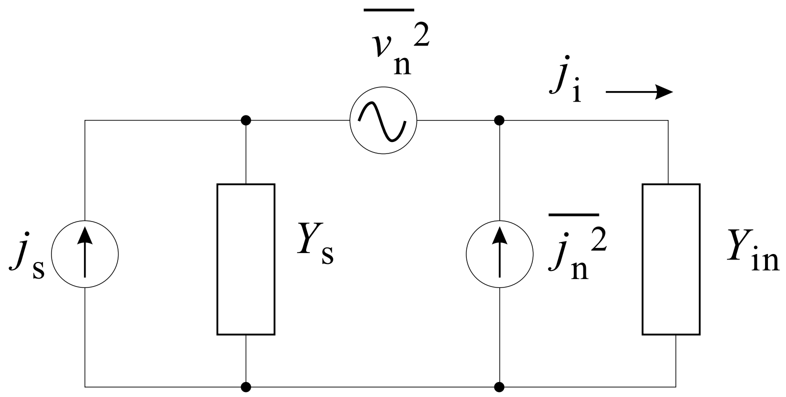


Figure 2: Simplified circuit for the input part with amplifier noise sources.

In the above simplified circuit,

$$j_{\rm i}=\frac{Y_{\rm in}}{Y_{\rm in}+Y_{\rm s}}j_{\rm s}+\frac{Y_{\rm in}}{Y_{\rm in}+Y_{\rm s}}j_n+\frac{1}{Z_{\rm in}+Z_{\rm s}}v_{\rm n}=\frac{Y_{\rm in}}{Y_{\rm in}+Y_{\rm s}}j_{\rm s}+\frac{Y_{\rm in}}{Y_{\rm in}+Y_{\rm s}}j_n+\frac{Y_{\rm in}Y_{\rm s}}{Y_{\rm in}+Y_{\rm s}}v_n$$

$$\therefore j_{\rm i}=\frac{Y_{\rm s}}{Y_{\rm in}+Y_{\rm s}}(j_{\rm s}+j_{\rm n}+Y_{\rm s}v_{\rm n}). \tag{8}$$

Henceforth we we the following notation; Y: admittance (generally complex), Z: impedance (generally complex), G: conductance (real part of admittance), B: reactance (imaginary part of admittance), R: resistance (real part of impedance); index s: source, index n: noise source, index u: uncorrelated, index c: correlation. Now the noise figure F is written as

$$F=\frac{\overline{j_{\rm i}^2}}{\overline{j_{\rm i}^2}(j_{\rm n}=0,v_{\rm n}=0)}=1+\frac{\overline{(j_{\rm n}+Y_{\rm s}v_{\rm n})^2}}{\overline{j_{\rm s}^2}}. \tag{9}$$

The current and the voltage noise sources in Fig.2 are separated for convenience but acutually there should be some correlation. We express this as

$$j_{\rm n}=j_{\rm u}+Y_{\rm c}v_{\rm n},$$

where $j_{\rm u}$ is uncorrelated part in the noise current and $Y_{\rm c}$ is the correlation admittance. Then F is written as

$$F=1+\frac{\overline{(j_{\rm u}+(Y_{\rm c}+Y_{\rm s}v_{\rm n})^2}}{\overline{j_{\rm s}^2}}=1+\frac{\overline{j_{\rm u}^2}+|Y_{\rm c}+Y_{\rm s}|^2\overline{v_{\rm n}^2}}{\overline{j_{\rm s}^2}}. \tag{10}$$

With the noise temperature T, the noise power spectra are written as $\overline{v_{\rm n}^2} = 4k_{\rm B}TR_{\rm n}$, $\overline{j_{\rm u}^2} = 4k_{\rm B}TG_{\rm n}$. Then

$$F = 1 + \frac{G_n^2 + |Y_{\rm c} + Y_{\rm s}|^2 R_{\rm n}^2}{G_{\rm s}^2} = 1 + \frac{G_{\rm n}}{G_{\rm s}} + \frac{R_{\rm n}}{G_{\rm s}}[(G_{\rm c} + G_{\rm s})^2 + (B_{\rm c} + B_{\rm s})^2). \tag{11}$$

In eq.(11), it is easy to see that $B_{\rm s} = -B_{\rm c}$ gives the minimum F as a function of $B_{\rm s}$. The condition of the minimum NF for $G_{\rm s}$ is obtained with equating $\partial F/\partial G_{\rm s}$ to 0 as

$$\frac{\partial F}{\partial G_{\rm s}} = -\frac{G_{\rm u}}{G_{\rm s}^2} + \frac{2R_{\rm n}}{G_{\rm s}}(G_{\rm c} + G_{\rm s}) - \frac{(G_{\rm c} + G_{\rm s})^2 R_{\rm n}}{G_{\rm s}^2} = 0, \tag{12}$$

which gives $R_{\rm n}G_{\rm s}^2 = G_{\rm u} + R_{\rm n}G_{\rm c}^2$. Then the solution is

$$G_{\rm s,opt} = \sqrt{G_{\rm c}^2 + \frac{G_{\rm u}}{R_{\rm n}}}, \quad B_{\rm s,opt} = -B_{\rm c}, \tag{13}$$

which gives the minimum value of F as

$$F_{\rm min} = 1 + 2G_{\rm c}R_{\rm n} + 2\sqrt{R_{\rm n}G_{\rm u} + (R_{\rm n}G_{\rm c})^2}. \tag{14}$$

With some algebra, one can reach

$$F = F_{\rm min} + \frac{R_{\rm n}}{G_{\rm s}}|Y_{\rm s} - Y_{\rm s,opt}|^2. \tag{15}$$

Interestingly in the field of electronics, sets of impedances (admittances) are often plotted on Smith (admittance) chart. Let us see how it looks for constant-F in (15). Remember, the reflection coefficient r at the end of a transmission line with characteristic impedance Z_0 terminated with impedance Z is

$$r = \frac{Z - Z_0}{Z + Z_0}. \tag{16}$$

Then generally admittance Y is expressed with characteristic admittance $Y_0 = Z_0^{-1}$ and r as

$$Y = \frac{1 - r}{1 + r}Y_0. \tag{17}$$

Then with

$$|Y_{\rm s} - Y_{\rm opt}|^2 = \left|\frac{1 - r_{\rm s}}{1 + r_{\rm s}} - \frac{1 - r_{\rm opt}}{1 + r_{\rm opt}}\right|^2 Y_0^2 = 4Y_0^2 \frac{|r_{\rm opt} - r_{\rm s}|^2}{|1 + r_{\rm s}|^2|1 + r_{\rm opt}|^2}, \tag{18}$$

and

$$G_{\rm s} = {\rm Re}(Y_{\rm s}) = Y_0{\rm Re}\left(\frac{1 - r_{\rm s}}{1 + r_{\rm s}}\right) = Y_0{\rm Re}\left[\frac{(1 - r_{\rm s})(1 + r_{\rm s}^*)}{(1 + r_{\rm s})(1 + r_{\rm s}^*)}\right] = Y_0{\rm Re}\left[\frac{1 - |r_{\rm s}|^2 - 2{\rm Im}(r)}{|1 + r_{\rm s}|^2}\right]$$
$$= Y_0\frac{1 - |r_{\rm s}|^2}{|1 + r_{\rm s}|^2},$$

eq.(15) is experssed as

$$\left|r_{\rm s} - \frac{r_{\rm opt}}{N + 1}\right| = \frac{\sqrt{N(N + 1 - |r_{\rm opt}|^2)}}{N + 1}, \tag{19}$$

where

$$N \equiv \frac{F - F_{\rm min}}{4R_{\rm n}Y_0}|1 + r_{\rm opt}|^2. \tag{20}$$

Equation (19) expresses a circle with the center and the radius as

$$\frac{r_{\rm opt}}{N + 1}, \quad \frac{\sqrt{N(N + 1 - |r_{\rm opt}|^2)}}{N + 1}. \tag{21}$$

4 Amplifier gain circles and practical design

Noise matching is desirable but not easy. Minimum NF tuning is comparatively easy but because the condition is different from power matching in general, it may sometimes cause even degradation of the signal. Hence some compromise is usually required.

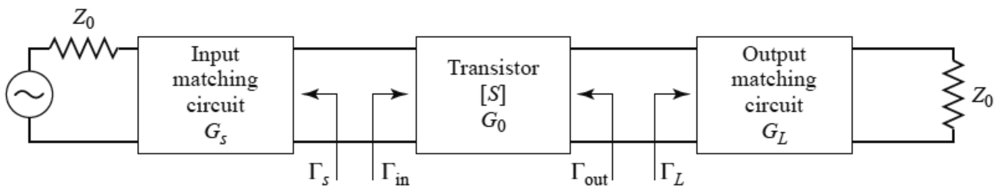


Figure 3: Setup of parameters to discuss the amplifier stability in wave parameters.

4.1 Stability requirements

High frequency transmission is often associated with reflections and they may work as a kind of feedback. Then we need to pay attention to the stability of the circuit. Let G_s, G_0 and G_L be gains for the input matching circuit, the transistor and the output matching circuit respectively as in Fig.3. And the complex reflection coefficients Γ's are also defined in Fig.3.

From the definition, it is easy to see that the "gains" are written as[3]

$$G_\mathrm{s} = \frac{1-|\Gamma_\mathrm{s}|^2}{|1-\Gamma_\mathrm{s}\Gamma_\mathrm{in}|^2}, \tag{22a}$$

$$G_0 = |S_{21}|^2, \tag{22b}$$

$$G_\mathrm{l} = \frac{1-|\Gamma_\mathrm{l}|^2}{|1-S_{22}\Gamma_\mathrm{L}|^2}. \tag{22c}$$

The total gain is $G_\mathrm{t} = G_\mathrm{s}G_0G_\mathrm{L}$.

The instability in the circuit in Fig.3 may occur when the imput or ouput impecance has a negative real part and cosequently $|\Gamma_\mathrm{in}| > 1$ or $|\Gamma_\mathrm{out}| > 1$.The conditions are inevitably frequency dependent. And in some cases, they depend also on load or source impedance (contitional stability. To be more rigorous, we need to consider the Nyquist (or other) stability criterion to avoid poles in the right half plane of transfer function. In the following we consider some simple cases.

The conditions $|\Gamma_\mathrm{in}| < 1$ and $|\Gamma_\mathrm{out}| < 1$ are written as

$$|\Gamma_\mathrm{in}| = \left|S_{11} + \frac{S_{12}S_{21}\Gamma_\mathrm{l}}{1-S_{22}\Gamma_\mathrm{l}}\right| < 1, \tag{23a}$$

$$|\Gamma_\mathrm{out}| = \left|S_{22} + \frac{S_{12}S_{21}\Gamma_\mathrm{l}}{1-S_{11}\Gamma_\mathrm{s}}\right| < 1. \tag{23b}$$

The boundary $|\Gamma_\mathrm{in}| = 1$ can, from (23a), be written as

$$|S_{11}(1-S_{22}\Gamma_\mathrm{L}) + S_{12}S_{21}\Gamma_\mathrm{L}| = |1-S_{22}\Gamma_\mathrm{L}|. \tag{24}$$

We write $\mathrm{det}S = S_{11}S_{12} - S_{12}S_{21}$ as Δ, then the above is

$$|S_{11} - \Delta\Gamma_\mathrm{L}| = |1-S_{22}\Gamma_\mathrm{L}|. \tag{25}$$

With squaring both sides and some algebra,

$$\Gamma_\mathrm{L}\Gamma_\mathrm{L}^* - \frac{(S_{22}-\Delta S_{11}^*)\Gamma_\mathrm{L} + (S_{22}^*-\Delta^* S_{11})\Gamma_\mathrm{L}^*}{|S_{22}|^2-|\Delta|^2} = \frac{|S_{11}|^2-1}{|S_{22}|^2-|\Delta|^2}. \tag{26}$$

Further we complete the square by adding $|S_{22}-\Delta S_{11}^*|^2/(|S_{22}|^2-|\Delta|^2)^2$ to obtain

$$\left|\Gamma_\mathrm{L} - \frac{(S_{22}-\Delta S_{11})^*}{|S_{22}|^2-|\Delta|^2}\right| = \left|\frac{S_{12}S_{21}}{|S_{22}|^2-|\Delta|^2}\right|. \tag{27}$$

This means the boundary of Γ_L on the complex plane is a circle (**stability circle**) with a center C_L and radius R_L, where

$$C_\mathrm{L} = \frac{(S_{22}-\Delta S_{11}^*)^*}{|S_{22}|^2-|\Delta|^2}, \quad R_\mathrm{L} = \left|\frac{S_{12}S_{21}}{|S_{22}|^2-|\Delta|^2}\right|. \tag{28}$$

Apparently we have symmetric result for the boundary of Γ_s as

$$C_\mathrm{s} = \frac{(S_{11}-\Delta S_{22}^*)^*}{|S_{11}|^2-|\Delta|^2}, \quad R_\mathrm{s} = \left|\frac{S_{12}S_{21}}{|S_{11}|^2-|\Delta|^2}\right| \tag{29}$$

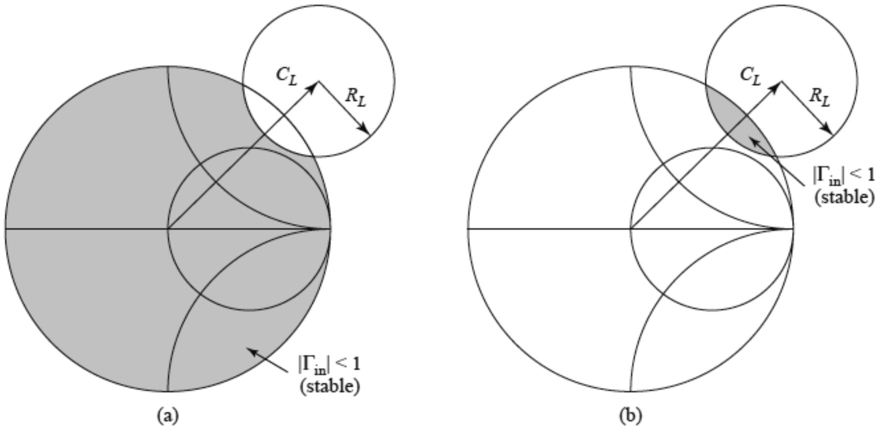


Figure 4: Output stability circles for a conditionally stable device. (a) $|S_{11}| < 1$. (b) $|S_{11}| > 1$.

for the center and the radius of the stability circle.

Figure 4 shows an example of a conditionally stable cases for $|S_{11}| < 1$ and $|S_{11}| > 1$. The latter may be possible when the device amplifies the reflection wave. Here we set $\Gamma_{\rm L} = 0$ with impecance matching. Then the boundary is $|\Gamma_{\rm in}| = |S_{11}|$. Then the shaded region in Fig.4(a) is stable. The same for $|S_{11}| > 1$ as shown in Fig.4.

References

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[3] D. M. Pozar, "Microwave Engineering" 4th ed., (Wiley, 2011).