電子回路論第10回
Electric Circuits for Physicists #10

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Outline

Power spectrum

Consider probability sets in the interval [0,*T*) with set index *j* .

$$x_j(t) = \sum_{n=1}^{\infty}(a_{jn}\cos\omega_n t + b_{jn}\sin\omega_n t), \quad \omega_n = \frac{2n\pi}{T}$$

$$\mathscr{P}_{jn} = (a_{nj}\cos\omega_n t + b_{nj}\sin\omega_n t)^2$$

(Power)

$$\langle \mathscr{P}_n \rangle = \frac{1}{2}\langle a_n^2 + b_n^2 \rangle$$

∵ cross product terms are averaged out

Random process:
Gaussian distribution in time

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{x^2}{2\sigma^2}\right]$$

$$\overline{(\delta x)^2} = \sigma^2, \quad \overline{\left(\sum_{j=1}^{m}\delta x_j\right)^2} = m\sigma^2$$

Then $\overline{\langle \mathscr{P}_n \rangle} = \sigma_n^2$

Power spectrum $\boldsymbol{G(\omega)}$

Frequency band width $\delta\omega$: separation between two adjacent frequencies

Power spectrum

Power spectrum $G(\omega)$

Frequency band width $\delta\omega$: separation between two adjacent frequencies

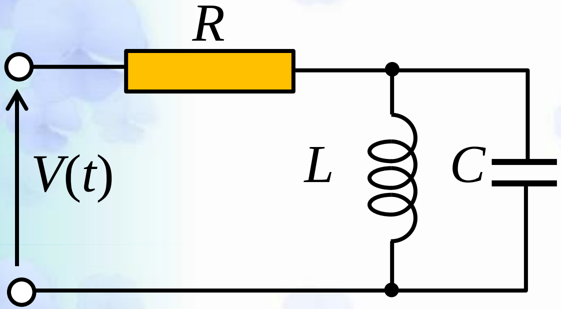
$$\delta\omega = \omega_{n+1} - \omega_n = \frac{2(n+1)\pi}{T} - \frac{2n\pi}{T} = \frac{2\pi}{T}$$

$$G(\omega_n)\frac{\delta\omega}{2\pi} = \overline{\langle \mathscr{P}_n \rangle}\ (= \sigma_n^2)$$

$$\overline{\langle x^2(t) \rangle} = \sum_{n=1}^{\infty} \overline{\langle \mathscr{P}_n \rangle} \qquad (\overline{x(t)} = 0)$$

$$= \sum_n G(\omega_n)\frac{\delta\omega}{2\pi} \to \int_0^\infty G(\omega)\frac{d\omega}{2\pi}$$

Fluctuation-dissipation theorem in the language of circuit



$$\omega_0 \equiv 1/\sqrt{LC}$$

$$Z(i\omega) = \frac{R(\omega_0^2 - \omega^2) + i\omega_0^2\omega L}{\omega_0^2 - \omega^2},$$

$$Y(i\omega) = \frac{\omega_0^2 - \omega^2}{R(\omega_0^2 - \omega^2) + i\omega_0^2\omega L}$$

$V(t)$ noise power spectrum → $G_v(\omega)$

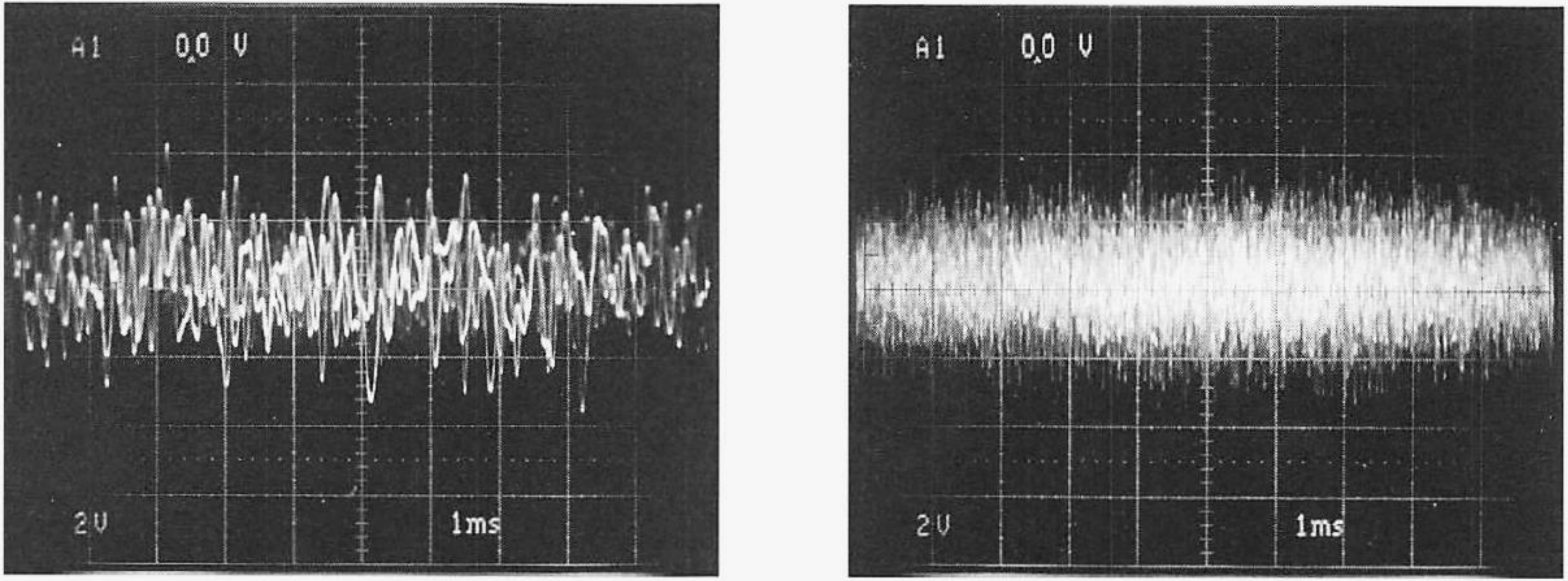
$$G_{\mathrm{v}}(\omega) = 4k_{\mathrm{B}}T\mathrm{Re}[Z(i\omega)]$$
$$= 4k_{\mathrm{B}}TR$$

Johnson-Nyquist noise
Thermal noise

White noise (noise with no frequency dependence) in the case of frequency independent resistance

One of the representations for the fluctuation-dissipation theorem

Thermal noise



(a) 上限周波数5 kHz (−3dB) 1 V_{rms}の熱雑音を1 ms/divで観測

(b) 上限周波数100 kHz (−3dB)1 V_{rms}の熱雑音を1 ms/divで観測

〈写真 1-1〉 熱雑音の測定

6.1.2 Wiener-Khintchine theorem

Autocorrelation function

$$
\begin{aligned}
C(\tau) &= \overline{\langle x(t)x(t+\tau)\rangle} \\
&= \overline{\sum_{n,m}\langle [a_n\cos\omega_n t + b_n\sin\omega_n t][a_m\cos\omega_m(t+\tau) + b_m\sin\omega_m(t+\tau)]\rangle} \\
&= \frac{1}{2}\sum_n \overline{\langle a_n^2 + b_n^2\rangle}\cos\omega_n\tau = \sum_n \overline{\langle \mathscr{P}_n\rangle}\cos\omega_n\tau \\
&= \int_0^\infty G(\omega)\cos\omega\tau\frac{d\omega}{2\pi}
\end{aligned}
$$

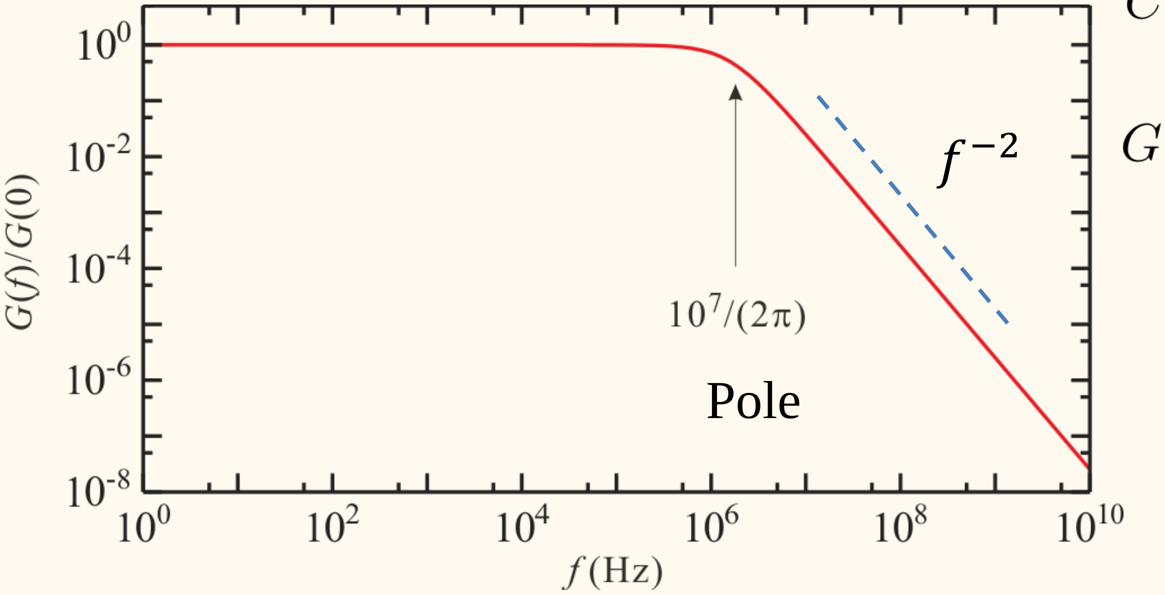
$$
C(\tau) = \int_0^\infty G(\omega)\cos\omega\tau\frac{d\omega}{2\pi}, \quad G(\omega) = 4\int_0^\infty C(\tau)\cos\omega\tau d\tau
$$

Wiener-Khintchine theorem

6.1.2 Wiener-Khintchine theorem

Example:

$\tau_0 = 10^{-7}\text{s} \quad (10\text{MHz})$



$$C(\tau) = \exp\left(-\frac{\tau}{\tau_0}\right)$$

$$G(f) = 4\int_0^\infty e^{-\tau/\tau_0}\cos(2\pi f\tau)d\tau$$

$$= \frac{4\tau_0}{1+(2\pi f\tau_0)^2}$$

“Unit” of noise

Noise: Power spectrum per frequency

$$\overline{j_n^2} = \overline{\delta J^2}/\Delta f, \quad \overline{e_n^2} = \overline{\delta V^2}/\Delta f$$

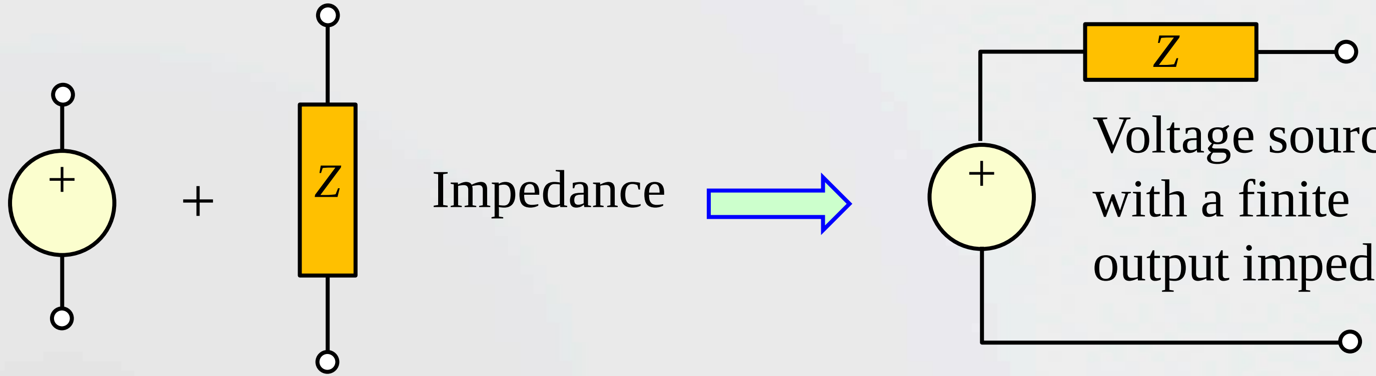
⬇ unit of $\sqrt{\overline{j_n^2}}, \quad \sqrt{\overline{e_n^2}}$

$$\mathrm{A}/\sqrt{\mathrm{Hz}}, \quad \mathrm{V}/\sqrt{\mathrm{Hz}}$$

6.1.3 Treatment of noise in electric circuits

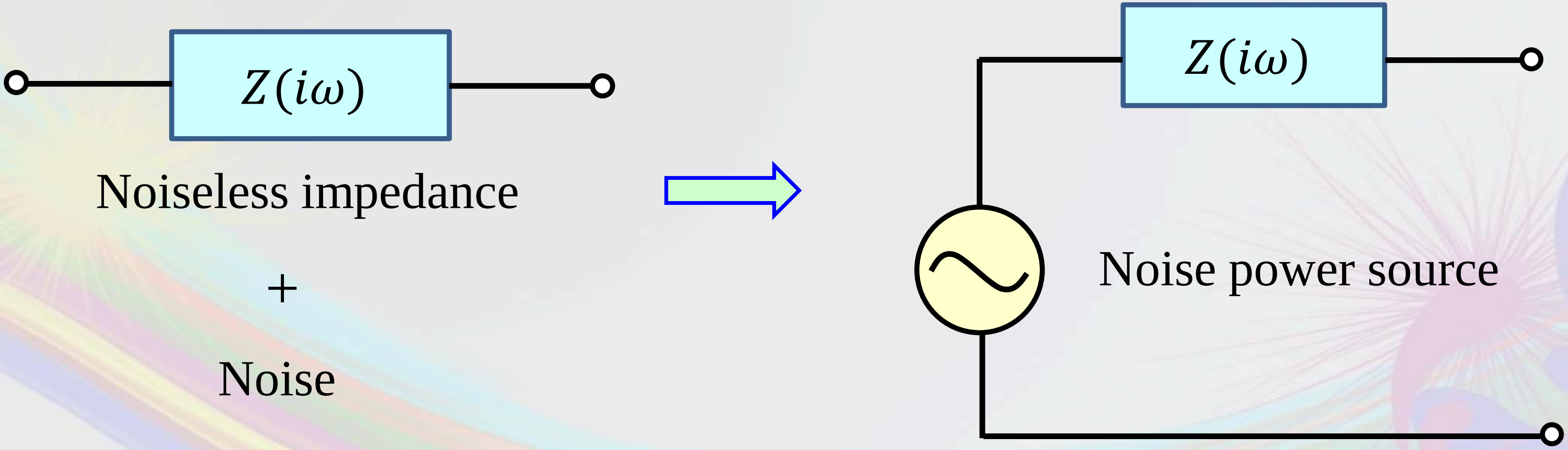
Remember:

Ideal voltage source



Impedance

Voltage source with a finite output impedance



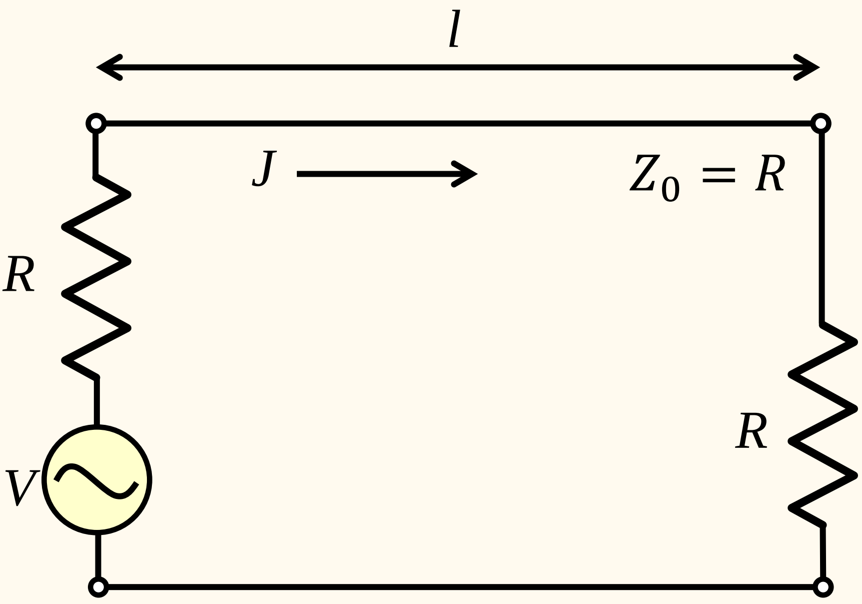
Noiseless impedance

+

Noise

Noise power source

6.1.4 Nyquist Theorem



Mode density on a transmission line with length l

$$\omega_n = c^* k_n = \frac{2\pi n c^*}{l} \quad \therefore \quad \delta\omega = \frac{2\pi c^*}{l}$$

Bidirectional → Freedom ×2

Bose distribution

$$f(\hbar\omega, T) = \frac{1}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1}$$

Thermal energy density per freedom

$$\frac{\hbar\omega}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1} \sim \frac{\hbar\omega}{1 + (\hbar\omega/k_{\mathrm{B}}T) - 1} = k_{\mathrm{B}}T \quad (k_{\mathrm{B}}T \gg \hbar\omega)$$

6.1.4 Nyquist Theorem

Thermal energy density in band $\Delta\omega$

$$2\frac{\Delta\omega}{\delta\omega}k_{\mathrm{B}}T = \frac{2k_{\mathrm{B}}Tl}{2\pi c^*}\Delta\omega$$, a half of which flows in one-direction

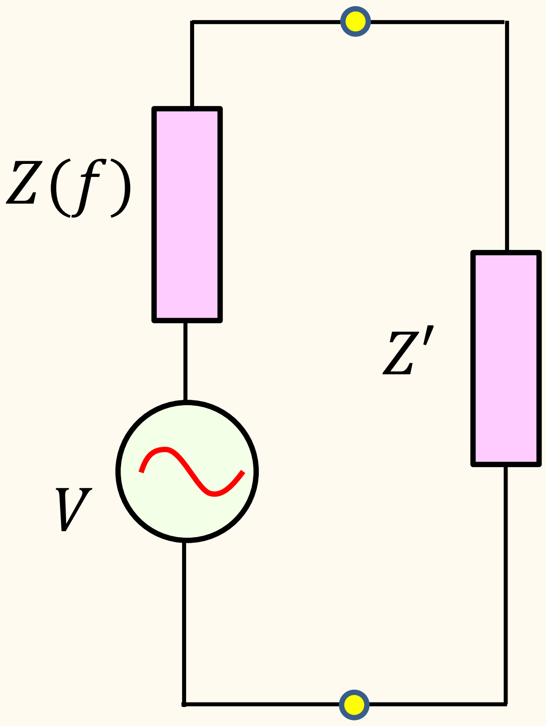
Energy flowing out from the end: $$\frac{k_{\mathrm{B}}Tl}{2\pi c^*}\Delta\omega \times \frac{1}{l} \times c^* = k_{\mathrm{B}}T\Delta f \quad (2\pi f = \omega)$$

equals the energy supplied from the noise source.

$$\overline{J^2}R = k_{\mathrm{B}}T\Delta f, \quad \overline{V^2} = 4Rk_{\mathrm{B}}T\Delta f$$

$$\sqrt{\overline{J^2V^2}} = 2k_{\mathrm{B}}T\Delta f$$ → Noise Temperature (primary thermometer)

6.1.4 Nyquist theorem



General impedance: $Z(f) = R(f) + iY(f)$

We assume that thermal noise energy per unit time is $k_B T\Delta f$ and consider the case in the left figure, in which Z' is matched to Z as

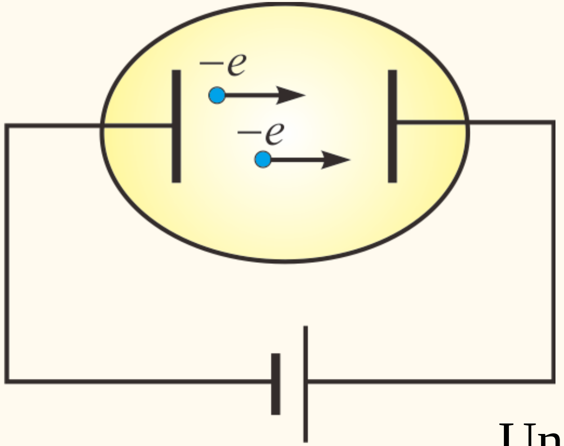
$$Z'(f) = Z^*(f) = R(f) - iY(f).$$

All the noise power is absorbed in Z'.
The total impedance is $2R(f)$. Then the thermal noise at the general impedance Z is

$$k_{\mathrm{B}}T\Delta f = \overline{J^2}(f)R(f) = \frac{\overline{V^2}(f)R(f)}{(2R(f))^2} = \frac{\overline{V^2}(f)}{4R(f)}$$

Voltage spectral density: $G(f) \equiv \dfrac{\overline{V^2}(f)}{\Delta f} = 4R(f)k_{\mathrm{B}}T$

6.1.5 Shot noise



Flow of single electron

Time domain: δ-function approximation

$$J_e(t) = e\delta(t - t_0)$$
$$= e\int_{-\infty}^{\infty} e^{2\pi i f(t-t_0)}df = 2e\int_{0}^{\infty} \cos\left[2\pi f(t-t_0)\right]df$$

Uniform 2*e* in frequency domain: fluctuation at each frequency

Coherent only at $t = t_0$

Current fluctuation density for infinitesimal band df $\delta J = d\sqrt{\langle J_e^2\rangle} = \frac{2e}{\sqrt{2}}df = \sqrt{2}edf$

Flow of double electron

$$\overline{\langle \delta j^2 \rangle} = (j_p + j_q e^{i\phi})(j_p + j_q e^{-i\phi}) = j_p^2 + j_q^2 + 2j_p j_q \cos\phi$$

ϕ: coherent phase shift → averaged out $\overline{\langle \delta j^2 \rangle} = j_p^2 + j_q^2 = 2 \times (\sqrt{2}e)^2 df$

6.1.5 Shot noise

Flow of *N*-electrons $$\overline{\langle \delta J^2 \rangle} = N \times 2e^2 df = 2e\overline{J} df \quad (\overline{J} = eN)$$

Quantum mechanical correlation → Modification from random (information!)

Example: pn junction

Current-Voltage characteristics: $$J(V) = J_0 \left[\exp\left(\frac{eV}{k_\mathrm{B}T} \right) - 1 \right]$$

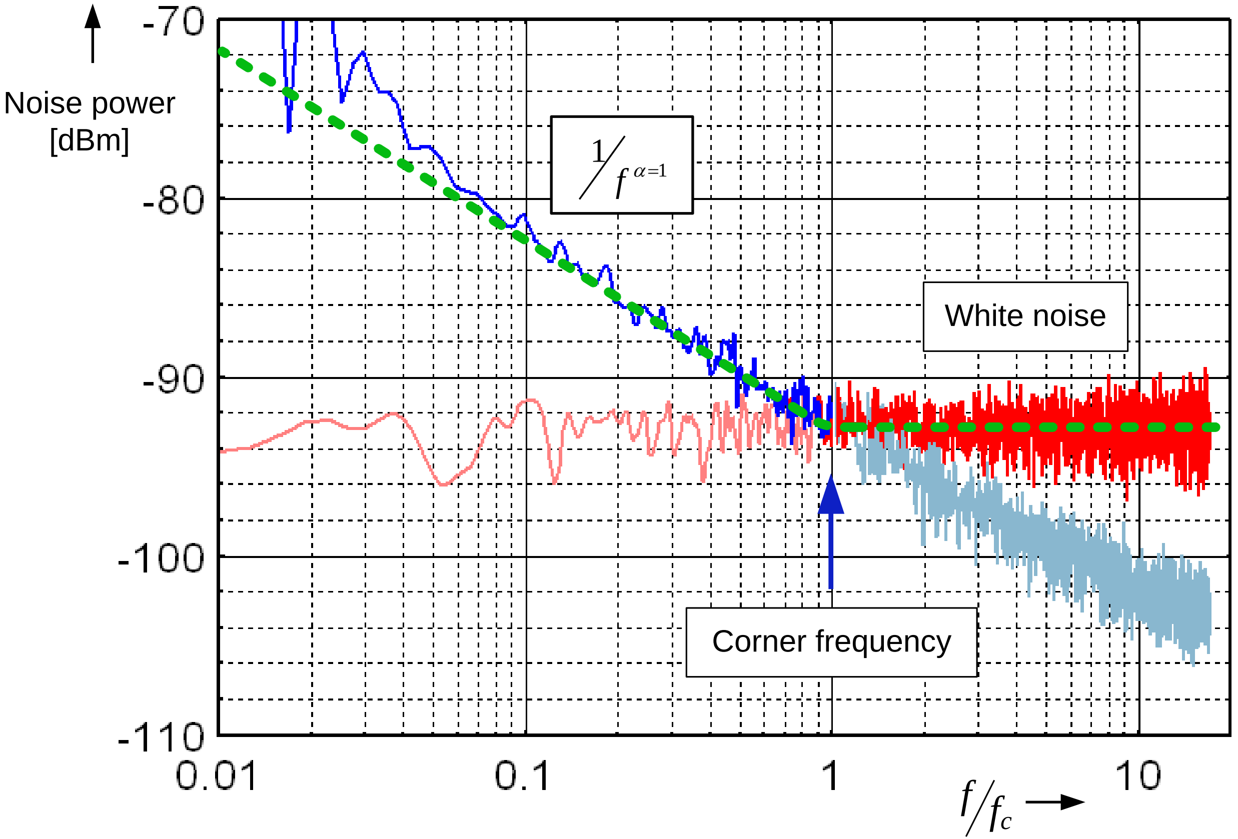
Differential resistance: $$r_\mathrm{d} = \left(\frac{dJ}{dV} \right)^{-1} = \left[\frac{eJ_0}{k_\mathrm{B}T} \exp\left(\frac{eV}{k_\mathrm{B}T} \right) \right]^{-1} = \frac{k_\mathrm{B}T}{e} \frac{1}{J + J_0}$$

$$J \gg J_0 \quad \rightarrow \quad r_\mathrm{d} \sim k_\mathrm{B}T/eJ$$

$$\overline{\langle (\delta J)^2 \rangle} = 2e \frac{k_\mathrm{B}T}{er_\mathrm{d}} df = 4k_\mathrm{B}T \frac{1}{2r_\mathrm{d}} df$$

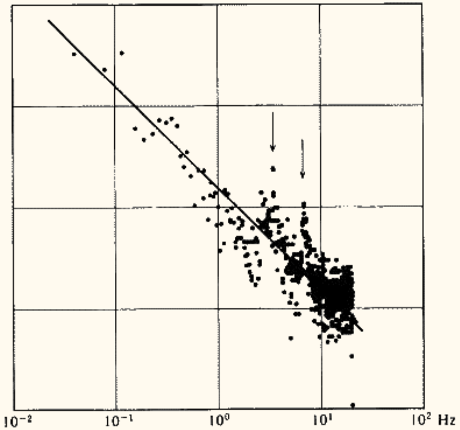
$$(\delta V)^2 = 4 \frac{r_\mathrm{d}}{2} k_\mathrm{B}T \Delta f$$

6.1.6 1/f noise

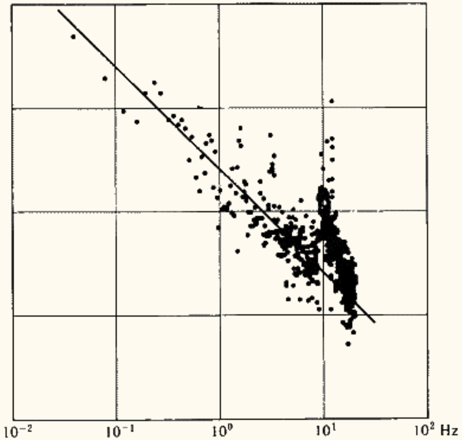


$$(\delta V)^2 = K J^a R^2 \frac{\Delta f}{f^\alpha}$$

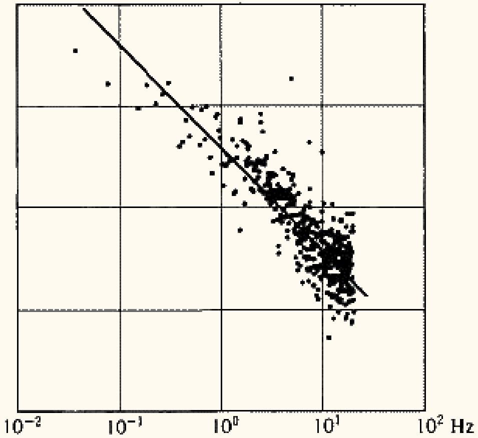
6.1.6 1/f noise



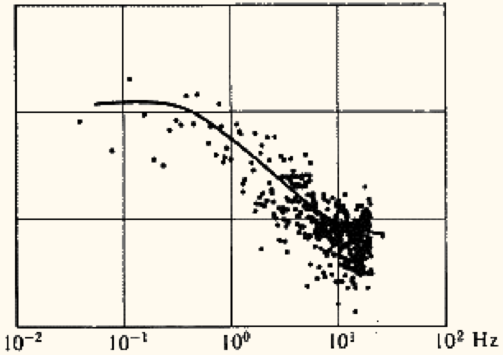
J. S. Bach, Brandenburg Concerto No.1



A. Vivaldi, Four Seasons, Spring



Kawai Naoko, Smile for me



S. Sato, Keshin (incarnation) II

リビング扇

F-CP324

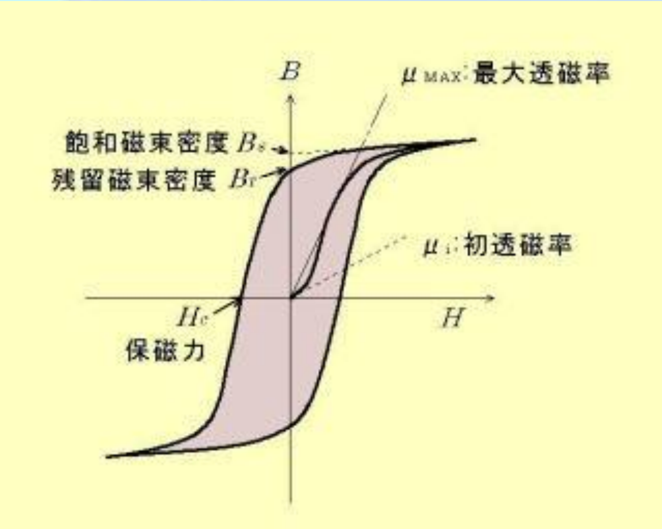
お部屋に広がる自然の心地いい風
「1/f ゆらぎ」「なめらか気流７枚羽根」。
首振り角度が90度で、よりワイド送風が可能に。
お子様のいるご家庭でも安心「チャイルドロックボタン」。

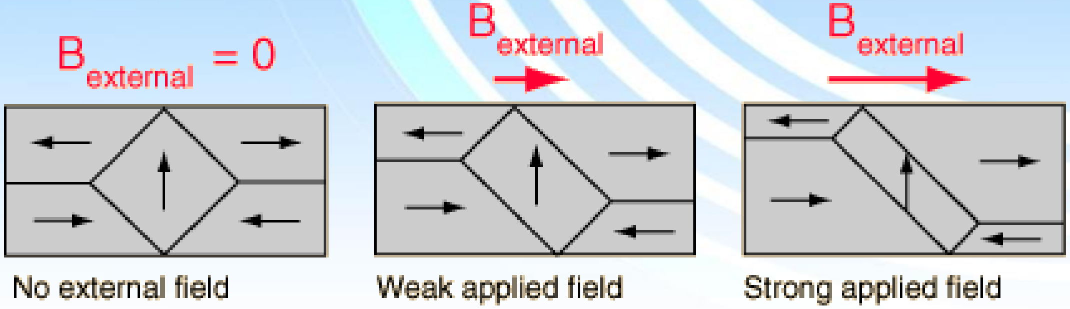
電源：AC100V (50Hz/60Hz)
消費電力(50Hz/60Hz)：38W/43W
羽根径：30cm(7枚羽根)
電源長さ：1.7m

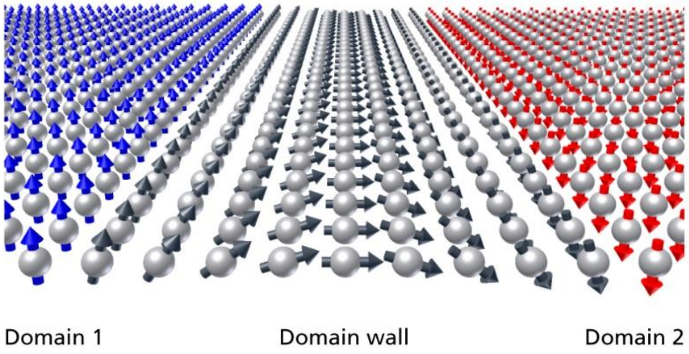
3段階風量切替・モコン付き・チャイルドロック機能
切タイマー：1/2/4時間

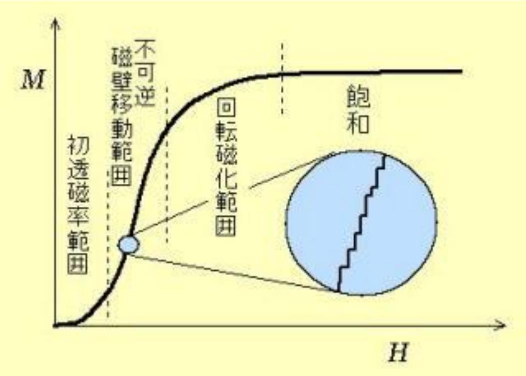
- ●首振り角度が90度で、よりワイド送風が可能に
- ●メーカー保証1年付き

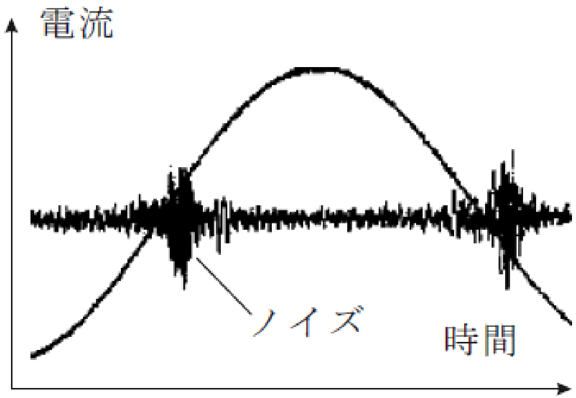
Other noises: Barkhausen noise





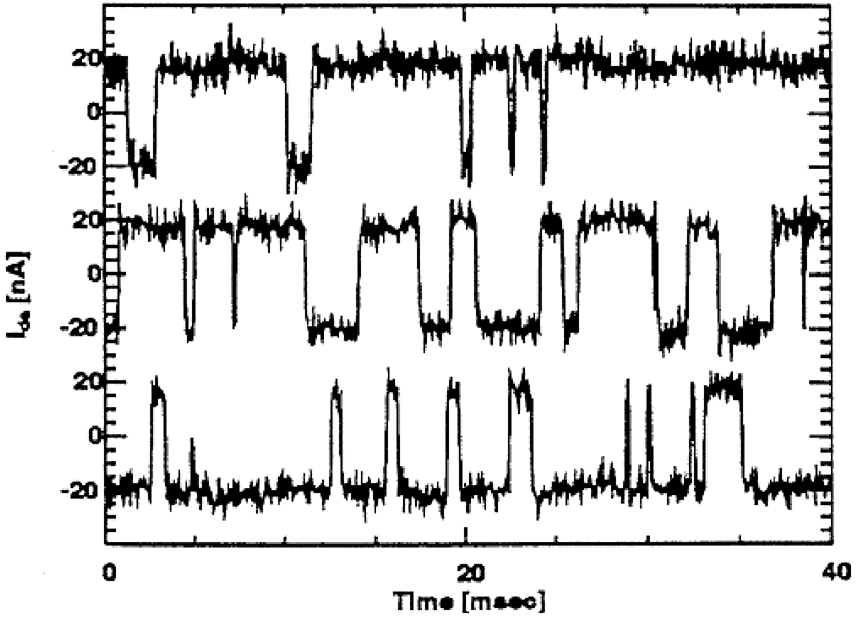


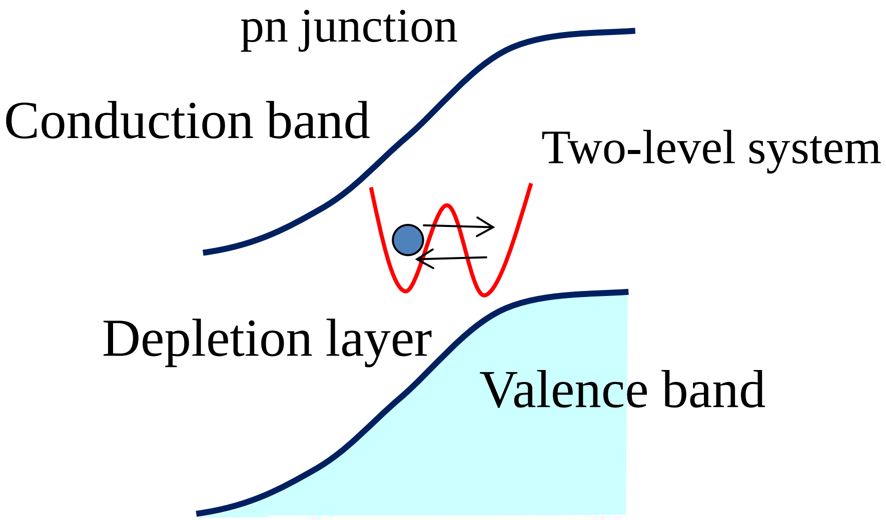




Other noises: popcorn noise

Popcorn noise, Burst noise





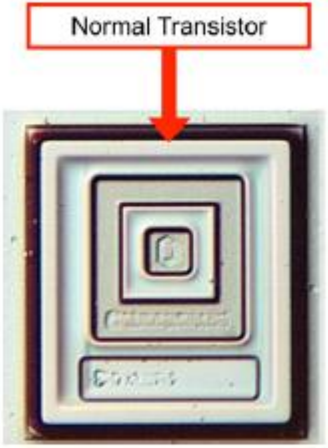
pn junction

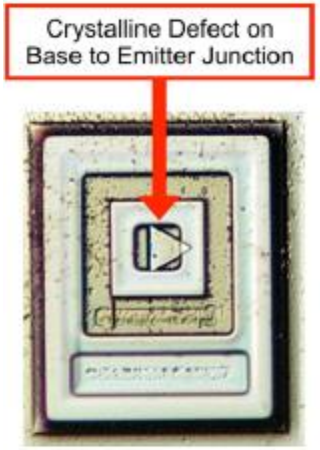
Conduction band

Two-level system

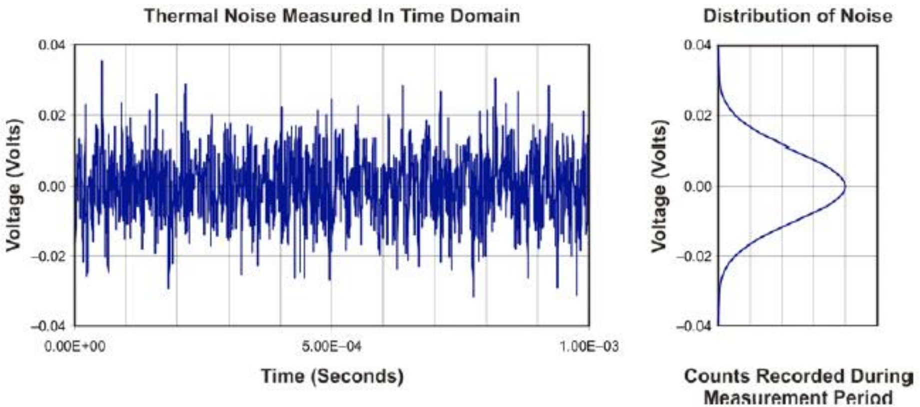
Depletion layer

Valence band

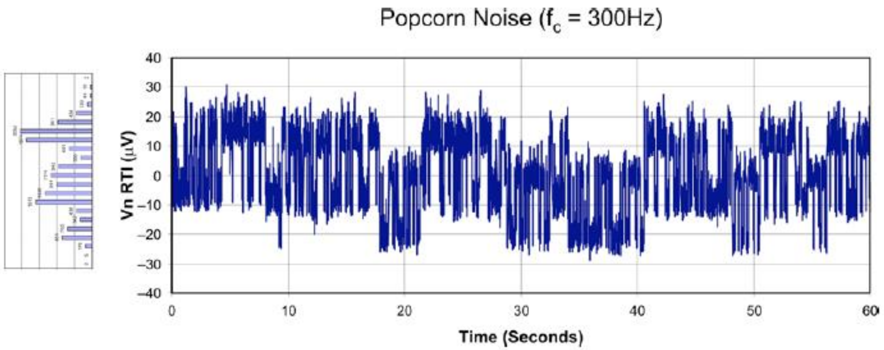




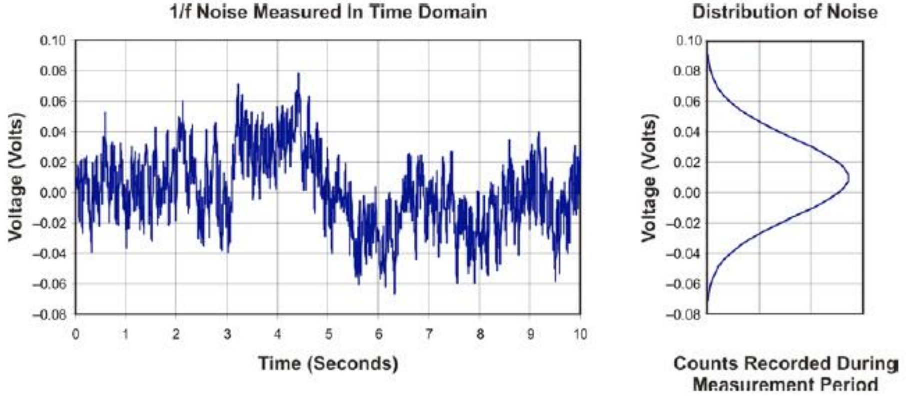
Probability distribution in popcorn noise



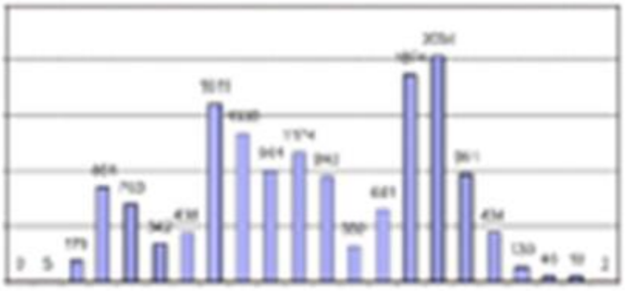
thermal noise



popcorn noise

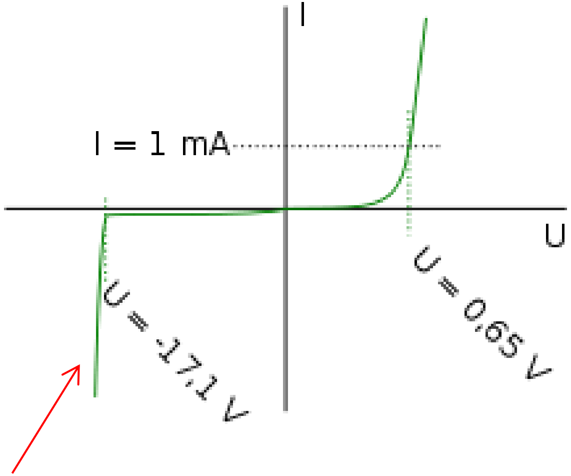


1/f noise

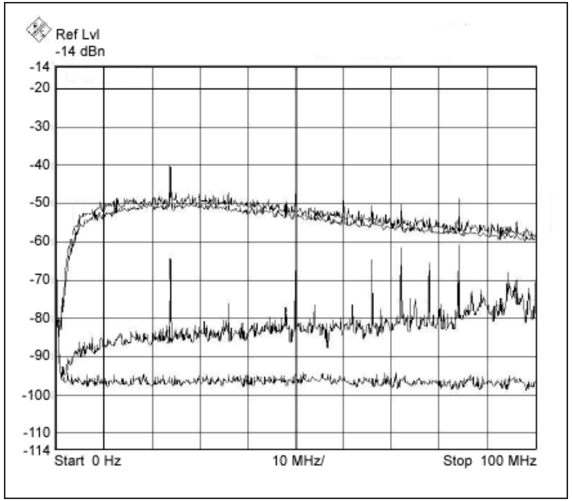


characteristic peaks in distribution

Avalanche noise

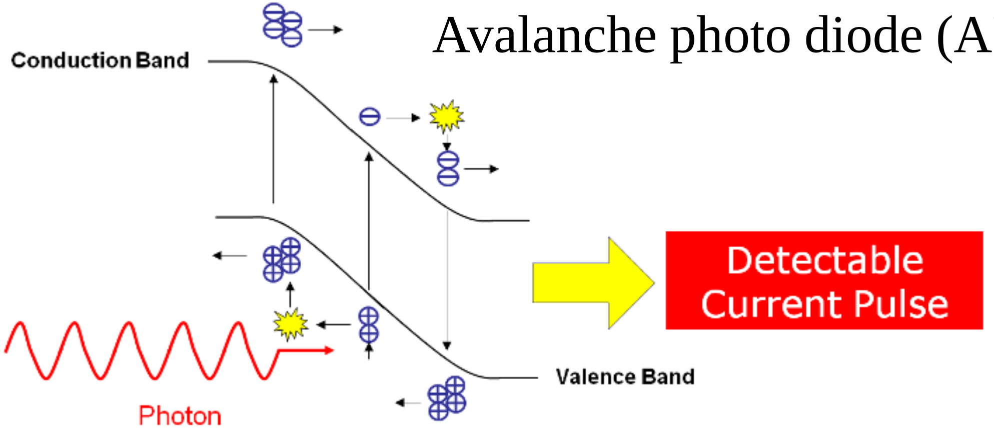


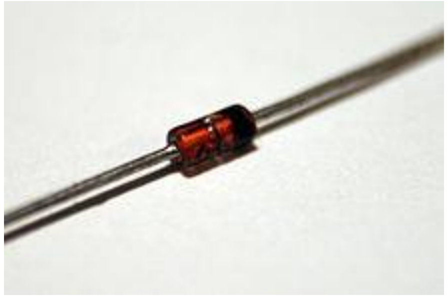
avalanche or Zener breakdown



White noise

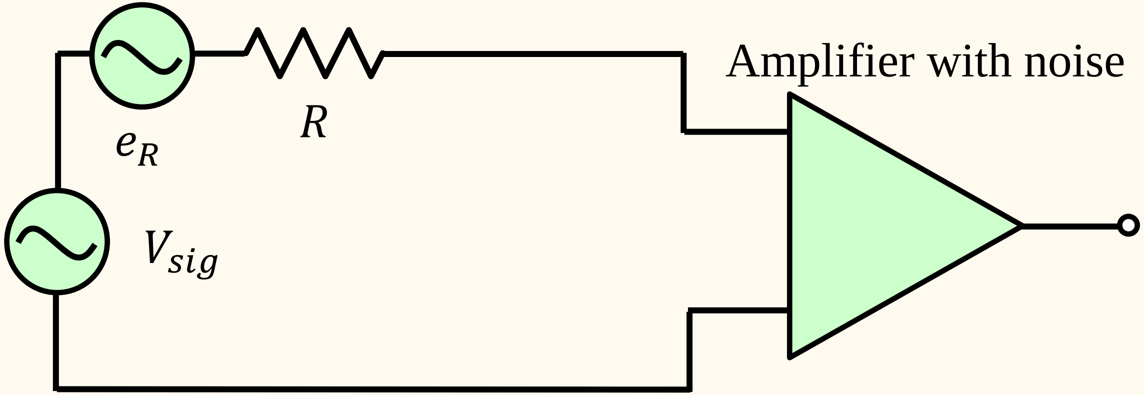
Avalanche photo diode (APD)



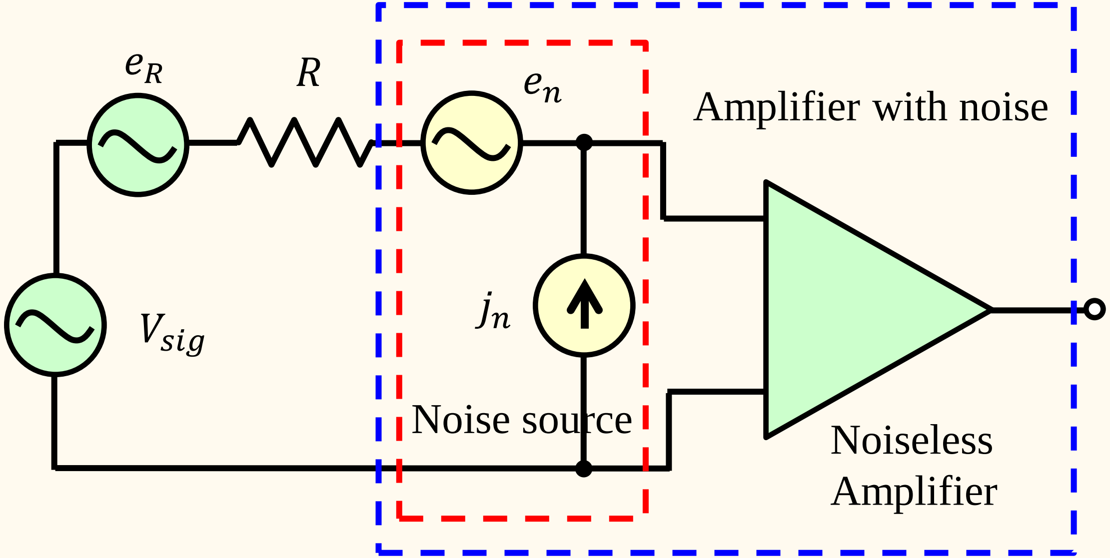


Zener voltage standard diode

6.2 Noise from amplifiers



Noise from the signal source can be expressed with a noise power source.
Then how for amplifiers ?



Noise from an amplifier also can be expressed with noise sources (voltage and current) at the input port.

6.2 Noise from amplifiers

Power gain G_p $\qquad e^2_{\mathrm{intotal}} = j_n^2 R^2 + e_{\mathrm{R}}^2 + e_n^2 = e^2_{\mathrm{out}}/G_{\mathrm{p}}$

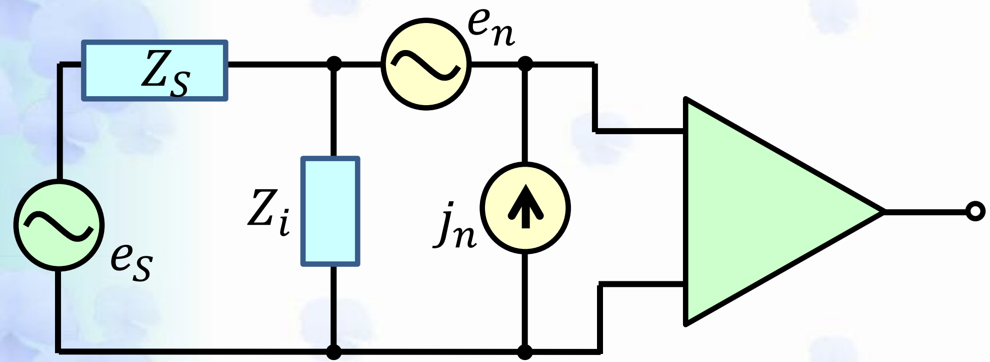
Signal to noise ratio: S/N ratio (power ratio)

Noise Figure: $\mathrm{NF} = 10\log_{10}\dfrac{(S/N)_{\mathrm{in}}}{(S/N)_{\mathrm{out}}} = 10\log_{10}\dfrac{S_{\mathrm{in}}N_{\mathrm{out}}}{S_{\mathrm{out}}N_{\mathrm{in}}}$

$$N_{\mathrm{out}} = G_{\mathrm{p}}\overline{e_{\mathrm{N}}^2}$$

$$\mathrm{NF} = 10\log_{10}\frac{S_{\mathrm{in}}G_{\mathrm{p}}\overline{e_{\mathrm{N}}^2}}{S_{\mathrm{in}}G_{\mathrm{p}}\overline{e_{\mathrm{R}}^2}} = 10\log_{10}\frac{\overline{e_{\mathrm{N}}^2}}{\overline{e_{\mathrm{R}}^2}} = 10\log_{10}\frac{\overline{e_n^2} + \overline{e_{\mathrm{R}}^2} + \overline{j_n^2}R^2}{\overline{e_{\mathrm{R}}^2}}$$

6.2.2 Noise (impedance) matching



What is noise matching?

Optimization of S/N ratio including the noise-source in the amplifier
(a care should be taken to the effect of noise to the object)

Strategy

Noises from the signal source, amplifiers: repel as much as possible
Signals from the source: absorb as much as possible

Noise temperature method (not almighty)

Nyquist theorem: $\sqrt{\overline{J^2}\,\overline{V^2}} = 2k_{\rm B}T\Delta f$

Noise temperature definition $(J(f), V(f))$

6.2.2 Noise (impedance) matching

Noise temperature and matched source impedance $$T_{\mathrm{a}} = \frac{\sqrt{\overline{e_n^2}\ \overline{j_n^2}}}{2k_{\mathrm{B}}}, \quad R_{\mathrm{bs}} = \sqrt{\frac{\overline{e_n^2}}{\overline{j_n^2}}}$$

Output noise temperature:

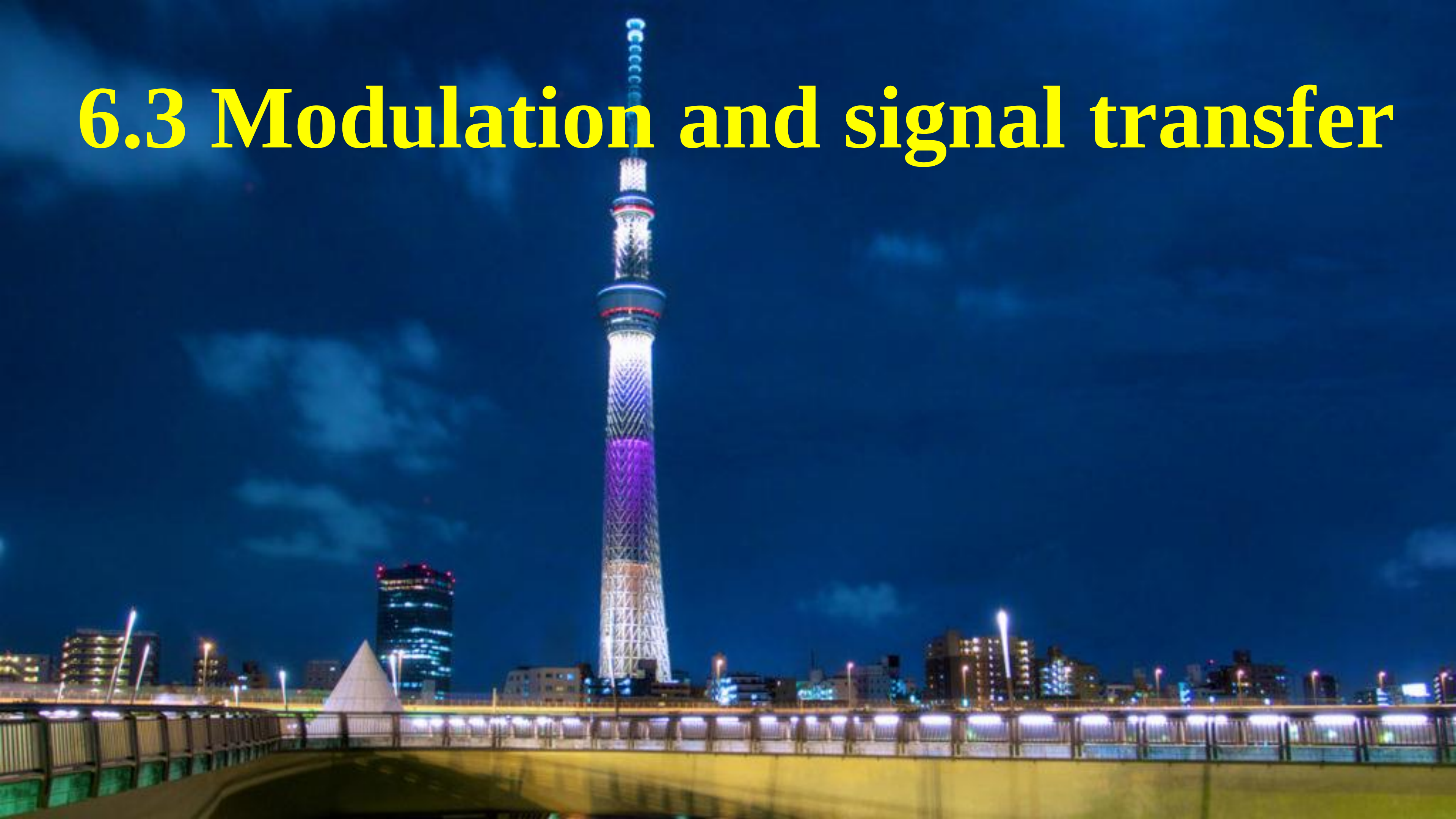
$$T_{\mathrm{n}} = \left(1 + \frac{\mathrm{Re}(1/Z_{\mathrm{i}})}{\mathrm{Re}(1/Z_{\mathrm{s}})}\right) \frac{T_{\mathrm{a}}}{2\mathrm{Re}Z} \left(\frac{|Z|^2}{R_{\mathrm{bs}}} + R_{\mathrm{bs}}\right), \quad \frac{1}{Z} \equiv \frac{1}{Z_{\mathrm{i}}} + \frac{1}{Z_{\mathrm{s}}}$$

Minimize T_n: $$Z_{\mathrm{i}} = \frac{1}{R_{\mathrm{bs}}^{-1} - Z_{\mathrm{s}}^{-1}}$$ Noise matching condition

$$T_{\mathrm{n}} = \left(1 + \frac{\mathrm{Re}(1/Z_{\mathrm{i}})}{\mathrm{Re}(1/Z_{\mathrm{s}})}\right) T_{\mathrm{a}}$$

See appendix for a practical design of noise matching circuit.

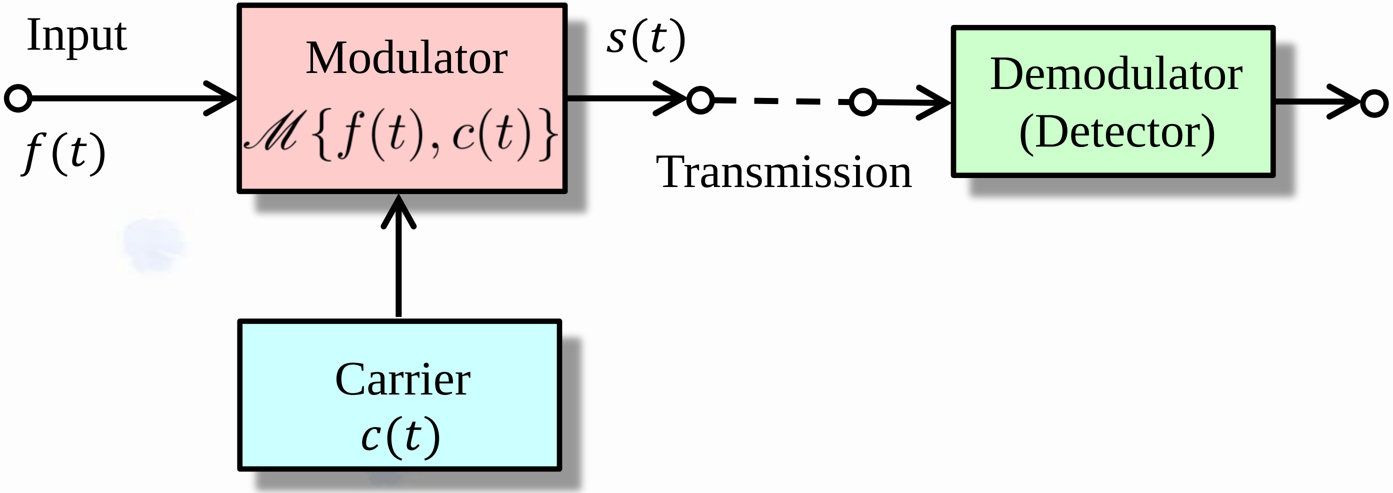
6.3 Modulation and signal transfer



6.3 Signal transmission

Electric communication
- Baseband communication
- Carrier communication

Carrier communication



Input

$f(t)$

Modulator

$\mathscr{M}\{f(t), c(t)\}$

$s(t)$

Transmission

Demodulator (Detector)

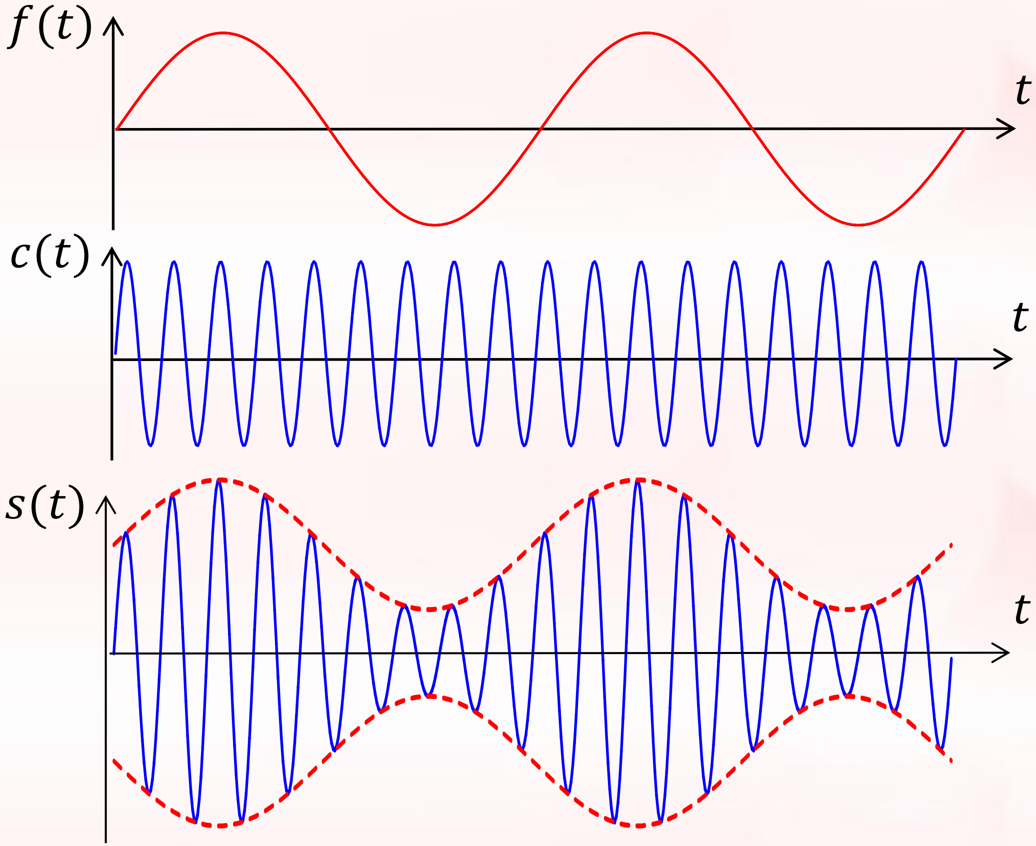
Carrier

$c(t)$

Modulation
- Amplitude modulation
- Frequency (Phase) modulation

Analog
Pulse

6.3.2 Amplitude modulation



$$c(t) = A\cos\omega_c t$$

$$s(t) = A[1 + mf(t)]\cos\omega_c t$$

m: Modulation index

$$0 < m \leq 1$$

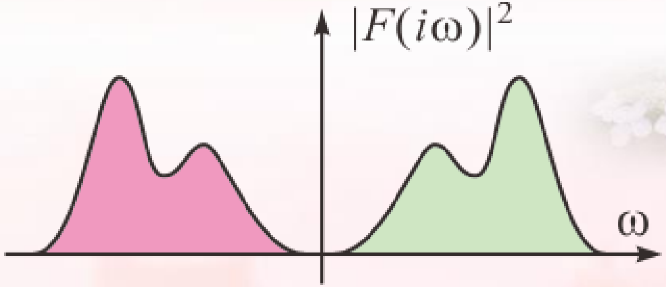
6.3.2 Amplitude modulation

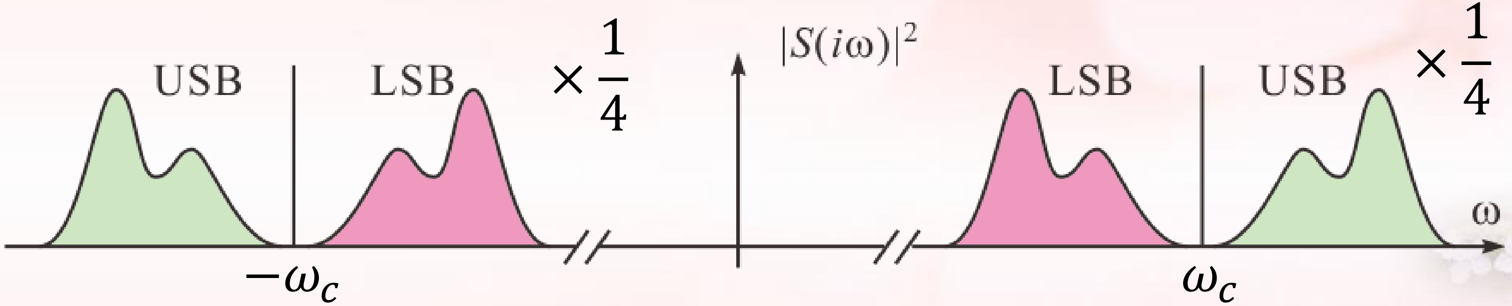
$$S(i\omega) = \int_{-\infty}^{\infty} s(t)e^{i\omega t}dt = \int_{-\infty}^{\infty} A[1 + mf(t)]\cos(\omega_c t)e^{i\omega t}dt$$

$$= A\left\{\pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] + \frac{m}{2}[F(i(\omega - \omega_c)) + F(i(\omega + \omega_c))]\right\}$$

$$F(i\omega) = \mathscr{F}\{f(t)\}$$

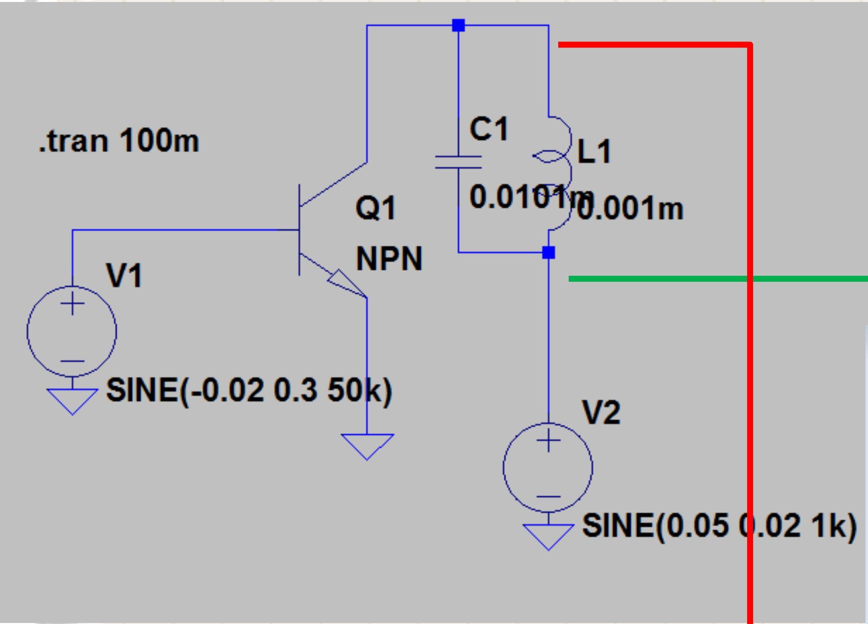
$f(t)$: Real $\quad F(i\omega) = F^*(-i\omega)$





Upper side band (USB), Lower side band (LSB)

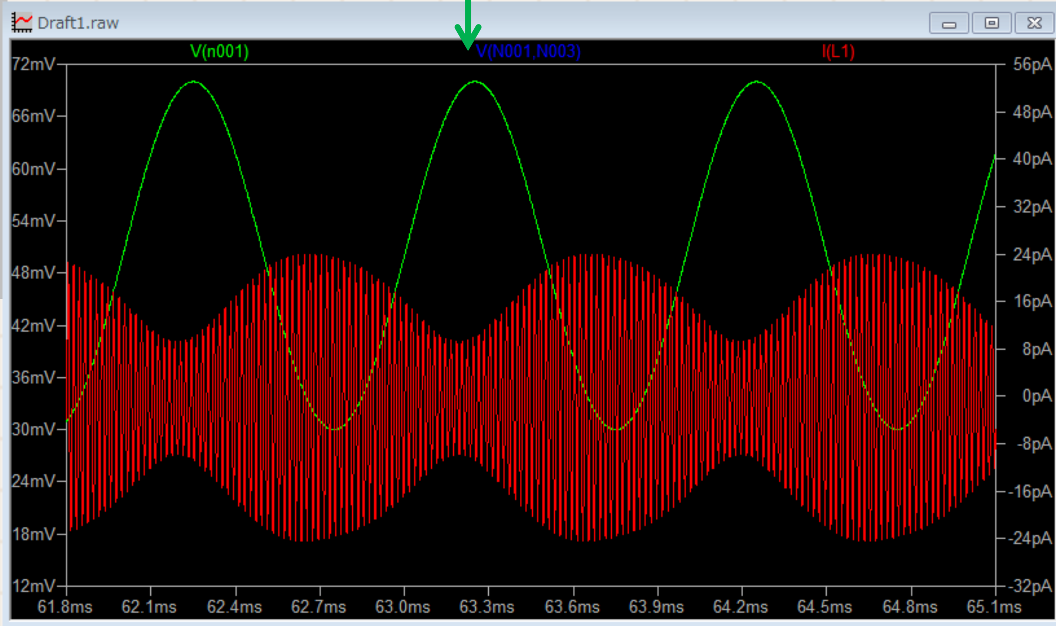
6.3.2 Amplitude modulation (circuit example)



Collector modulation circuit

C-class (non-linear) amplification region

Modulation voltage

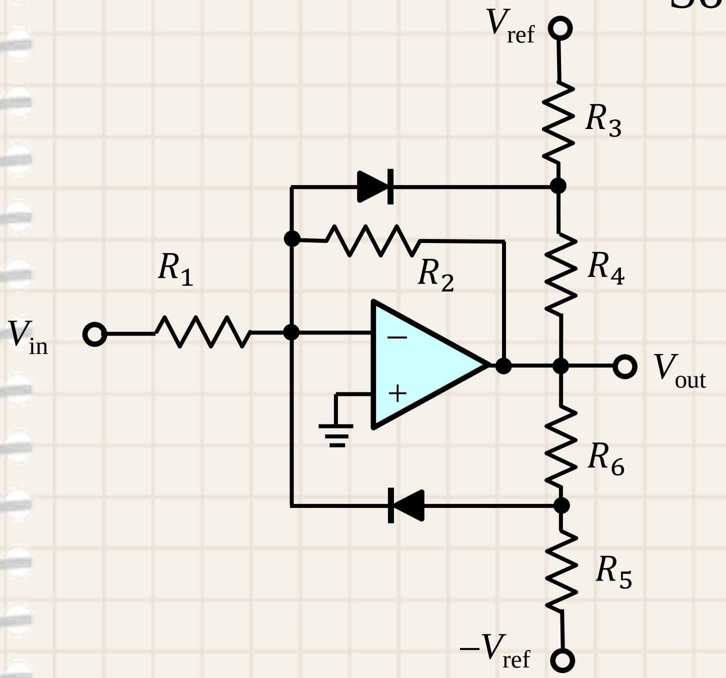


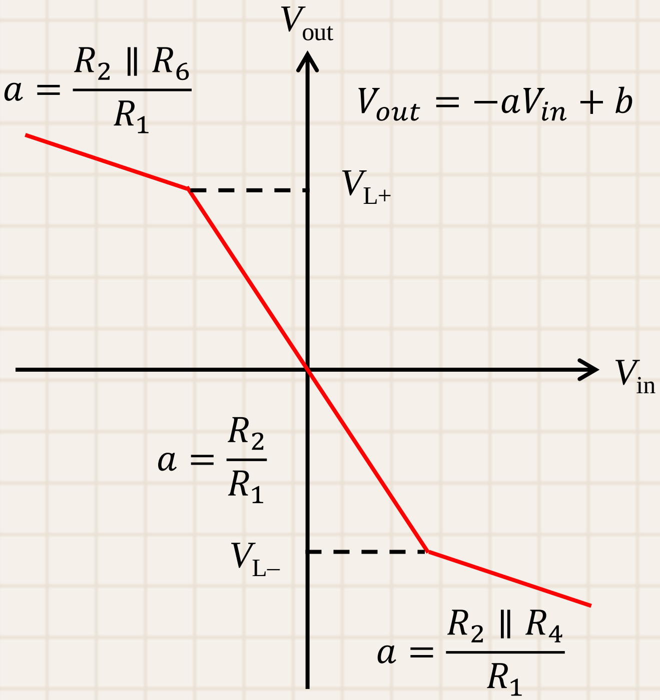
Current through the inductor

6.3.2 Amplitude modulation (circuit example 2)

Idea: Modulation of oscillator circuit

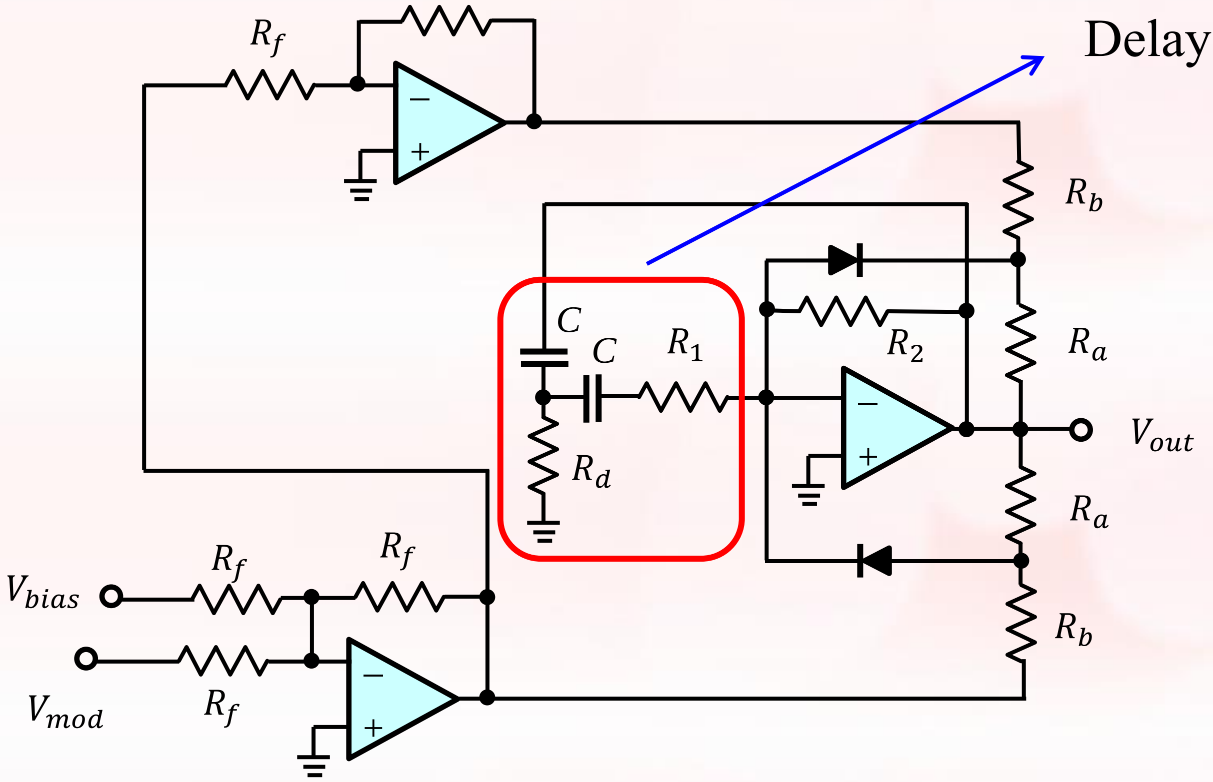
Soft limiter circuit





$$V_{L-} = -\frac{R_4}{R_3} V_{ref} - \left(1 + \frac{R_4}{R_3}\right) V_{th}$$: controllable with V_{ref}

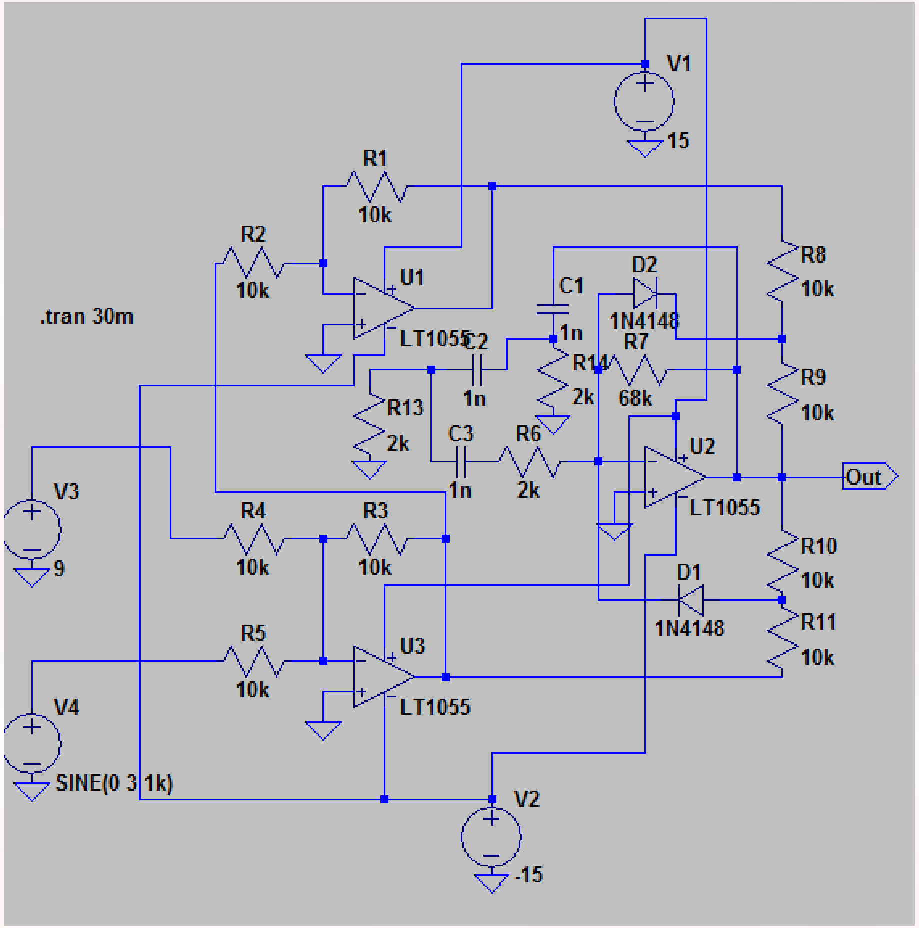
6.3.2 Amplitude modulation (circuit example 2)

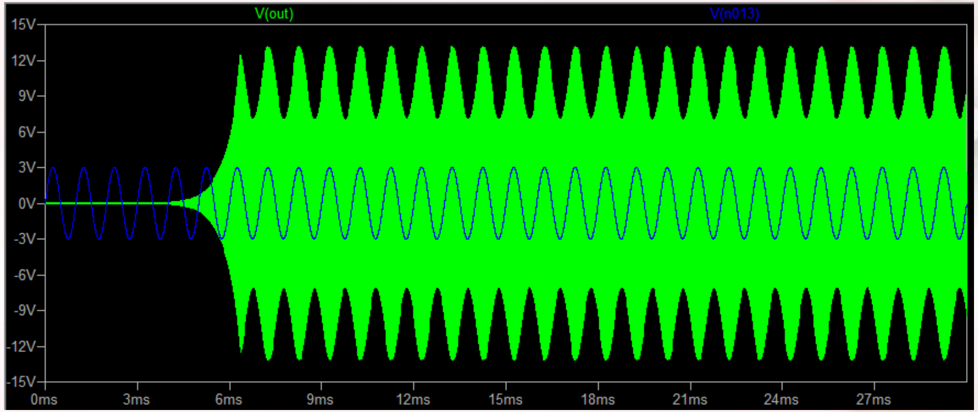


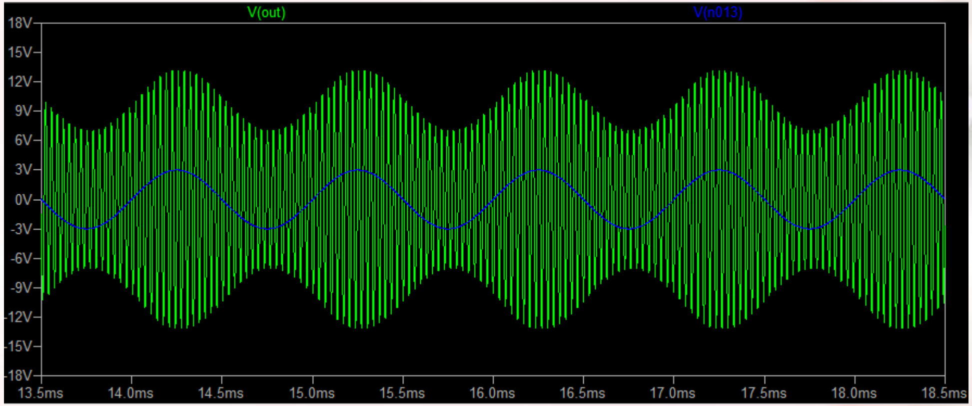
Delayed feedback → Oscillation

The amplitude is softly limited with the modulation voltage.

6.3.2 Amplitude modulation (circuit example 2)







6.3.2 Amplitude modulation (demodulation)