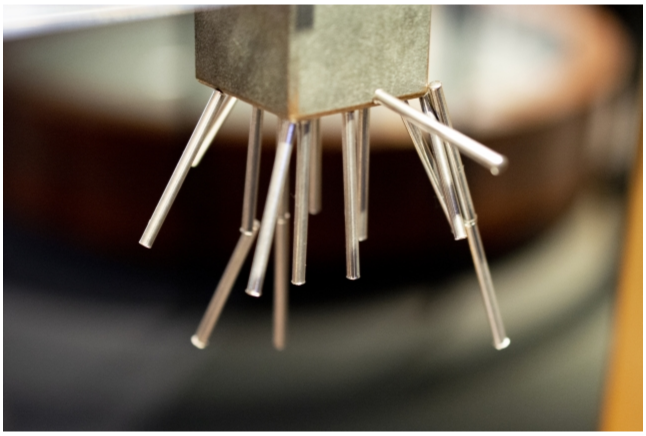
6th April (2022) Shingo Katsumoto, Institute for Solid State Physics, University of Tokyo

Chapter 1
Basic Notions of Magnetism



Magnetic fields are well known as “fields” that generate remote (but actually proximity) forces as well as magnetic fields or gravitational fields. Materials respond to magnetic fields on one hand, create magnetic fields on the other. Such properties are called “magnetism.” Perhaps the most prominent “magnetism” for us is spontaneous magnetization represented by permanent magnets. On the other hand, every material has some magnetic properties. Then what do we call magnetism? What is the origin of magnetism? We will consider these problems in this half-year lecture. I do not think I can give you sufficient answers though I would like to try to give you some usuful hints to consider the problems.

In this chapter, we will have a short look at very basic notions in magnetism. I may skip some of the contents in the lecture notes in the real lectures due to the time limitations.

1.1 Electromagnetic fields in the vacuum and those with materials

We skip the very elementary electromagnetism, with which all of you are already familiar. First, we consider the magnetic properties of matter phenomenologically.

1.1.1 The Maxwell equaitons and magnetic moment

In classical theory, the electromagnetic field in a vacuum is described by the Maxwell equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{1.1a}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{1.1b}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1.1c}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{j} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right). \tag{1.1d}$$

We adopt MKSA (SI) unit of system (Appendix 1A). The annotations of symbols may be skipped, but for confirmation, $\boldsymbol{B}$ is the magnetic flux density with unit of [T] (Tesla) in SI. The unit [T] is the same as [Wb/m^2], where we consider the number of magnetic flux [Wb] (Weber). The unit of magnetic field intensity is [A/m], and we usually use symbol $\boldsymbol{H}$ the quantity measured in this unit. In the vacuum, they are in a linear relation $\boldsymbol{B} = \mu_0 \boldsymbol{H}$ with the coefficient μ_0 (permeability of vacuum). Because μ_0 μ_0 has a physical dimension in MKSA unit system, thus $\boldsymbol{B}$ is a different quantity from $\boldsymbol{H}$ to be strict. However $\mu_0 = 4\pi \times 10^{-7}$ [H/m] is a universal scalar and we often call $\boldsymbol{B}$ as "magnetic field", as well as $\boldsymbol{H}$ in the vacuum. In materials, the situation changes.

In eq. (1.1c), $\boldsymbol{E}$ and $\boldsymbol{B}$ are not symmetrical even after tunings of coefficients. The origin is ρ (charge density) in the rhs of eq. (1.1a), and $\boldsymbol{j}$ (current density) in the rhs of eq. (1.1d). These come from the fact that the electric monopole exsits. Though the possibility of the existence of magnetic monopole is not completely eliminated, there has been no convincing report on the finding of magnetic monopole. At present, as in eqs. (1.1), we do not consider the existence of magnetic monopole. In eqs. (1.1), hence, magnetic fields are created by electric currents and time-derivatives of electric field as in eq. (1.1d). However, as we see later, electrons and some of nuclei have spin angular monments and associated magnetic dipole moments. These produce dipole magnetic fields around them.

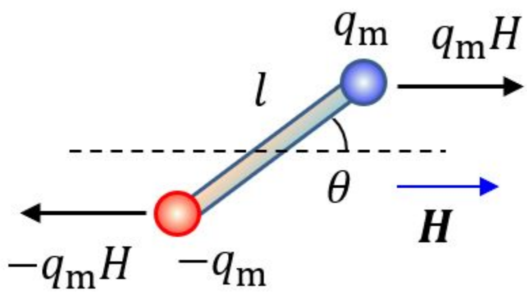


Fig. 1.1 Pair of force moment on a magnetic dipole in a magnetic field.

The concept of dipole magnetic field can be introduced as the shrinkage limit of a circular current(1B.2). On the other hand, in correspondence with the electric dipole, introduction of fictitious magnetic charges which always appear as a pair with the same amplitude and the opposite sign, and the limit of shrinkage under the condition of keeping the product (magnetic charge)×(distance) constant. Then let us define magnetic moment as follows. We use [Wb] (weber) as the unit of "magnetic charge" in MKSA system in parallel with the unit of electric charge ([C]) = that of electric flux due to the Gauss theorem. We consider a magnetic dipole (before taking the shrinkage limit), which has magnetic charges $\pm q_{\rm m}$ with distance l. Let the dipole be placed in a uniform magnetic field H and has an angle θ to the field (Fig. 1.1). The magnetic charges get the force $q_{\rm m} H$ from the field and the dipole get the pair of force

$$L = -q_{\rm m} l H \sin\theta = -(q_{\rm m} l/\mu_0) B \sin\theta. \tag{1.2}$$

The quantities that depend $q_{\rm m}$ and l only in the form of their product $q_{\rm m} l$, do not change with taking the limit. Hence we write

$$\mu \equiv q_{\rm m} l/\mu_0, \tag{1.3}$$

and call μ **magentic moment**.

Because the couple coment in eq. (1.2) drives a rotation of the magnetic moment to the direction of $\theta = 0$, it can be expressed by a static magnetic potential $\phi_{\rm m}$ as follows.

$$\phi_{\rm m} = -\mu B \cos\theta. \tag{1.4}$$

By generalizing the above to a vector representation, we get

$$\boldsymbol{L} = \boldsymbol{\mu} \times \boldsymbol{B}, \quad \phi_{\rm m} = -\boldsymbol{\mu} \cdot \boldsymbol{B}. \tag{1.5}$$

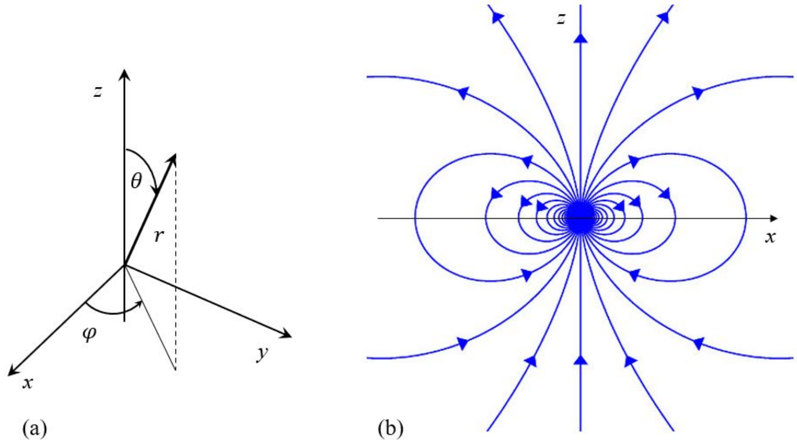


Fig. 1.2 (a) Schematic view of polar coordinate (r, θ, φ). (b) Schematic diagram of magnetic power force lines from magnetic dipole.

1.1.2 Dipole interaction

Let a magnetic moment along z-axis be at the origin. We use the polar coordinate (r, θ, φ) in Fig. 1.2(a).

From the spherical fields that the magnetic charges create, it is easy to see that the magnetic field has no component along φ. It is well known that the magnetic flux density (B_r, B_θ) along (r, θ) are

$$\left.\begin{aligned} B_r &= \frac{|\boldsymbol{\mu}|}{4\pi\mu_0}\frac{2\cos\theta}{r^3}, \\ B_\theta &= \frac{|\boldsymbol{\mu}|}{4\pi\mu_0}\frac{\sin\theta}{r^3}. \end{aligned}\right\} \tag{1.6}$$

We see they are inverse proportional to the cube of distance. The derivation can be fined in Appendix 1B1. We can draw the magnetic force line by connecting tangent lines of magnetic field vectors and obtain, e.g., that in Fig. 1.2(b)[*1] .

Next we consider two such magnetic moments $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$. As in Fig. 1.3, the vector going from $\boldsymbol{\mu}_1$ to $\boldsymbol{\mu}_2$ is written as $\boldsymbol{r}$. $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ form the potential in eq. (1.4) in the magnetic fields of eq. (1.6). Then the total potential of the two magnetic moments is

$$U = \frac{1}{4\pi\mu_0 r^3}\left\{\boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2 - \frac{3}{r^2}(\boldsymbol{\mu}_1\cdot\boldsymbol{r})(\boldsymbol{\mu}_2\cdot\boldsymbol{r})\right\}. \tag{1.7}$$

The derivation is given in Appendix 1B2. The interaction between the moments expressed by eq. (1.7) is called **dipole-dipole interation**. In the potential of eq. (1.7), the stable configuration can be obtained by maximizing the amplitude of the second term because the sign is minus and the coefficient is larger than the first one. $|r|$ is canceled by the denominator

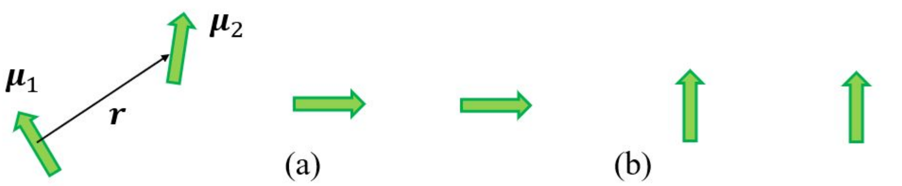


Fig. 1.3 Classical interaction of two magnetic dipoles. (a) Stable configuration. (b) Unstable configuration.

[*1] In this figure, it is not taken into consideration that the magnetic field line density is proportional to the magnetic field.

and the numerator, hence the stable configuration is obtained as in Fig. 1.3(a). On the ohter hand the configuration in Fig. 1.3(b) has a higher energy and is unstable. The effect of classical dipole-dipole interaction is generally much smaller than the quantum mechanical exchange interaction. However, in some characteristic phenomena, it plays important roles.

A more fundamental problem of multiple magnetic moments is how we can view a set of magnetic moments from a far distance. The problem of taking a limt of moment-moment approach with a constant moment-distance product, is the problem of **multipole**. However, simply looking from a distance, the problem is the same as that a set of various charges from a distance can be treated as a point charge with the sum of the charges. This from the linearity of the Maxwell equations. Then let $\{\boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \cdots\}$ be the magnetic moments under consideration and we see the set as a moment which is the sum

$$\boldsymbol{\mu} = \sum_i \boldsymbol{\mu}_i. \tag{1.8}$$

1.1.3 Magnetization of materials

Generally a magnetic field induces a magnetic moment in materials. This phenomenon is called magnetization of a material. Assuming a uniform distribution of induced magnetic moments, the moment per unit volume is called **magnetization** or **magnetic polarization**. Now, we write the magnetization as $\boldsymbol{M}$. Then from definition (1.3), we express the induced magnetic moment as the sum of equal descrete moment $\boldsymbol{\mu} = q_{\rm m}\boldsymbol{l}/\mu_0$ of concentration N (per unit cell) as

$$\boldsymbol{M} = \sum_{\rm unitvol.} \boldsymbol{\mu} = Nq_{\rm m}\boldsymbol{l}/\mu_0 \equiv \rho\boldsymbol{l}/\mu_0. \tag{1.9}$$

Here, $\rho \equiv Nq_{\rm m}$ is like a density of magnetic charges. In the naive model described in Fig. 1.4, small bar magnets with length $l = |\boldsymbol{l}|$ are aligned. The magnetic charges of neighboring magnets cancel each other due to the zero distance and no magnetic charge exists inside the material naturally. The magnetic charges then appear just at the ends of the material.

When the end surfaces are taken perpendicular to the magnetic polarization, the surface magnetic charge density is $q_{\rm m}$ times s, the areal density of “rods” which are the serieses of bar magnets. We consider a slab with unit area and height l, which should contain just one moment along the height, then the total number inside should be s. On the other hand from the definition the number should be Nl, which means $s = Nl$. Therefore the areal density of magnetic charge σ at the ends is given by

$$\sigma = q_{\rm m}s = q_{\rm m}Nl = \mu_0|\boldsymbol{M}|, \tag{1.10}$$

that is, the magnetic polarization is the same as the areal density of magnetic charges.

As considered above, the magnetic moment induced in the material (there is also spontaneous magnetization that occurs without an external magnetic field), that is, the magnetization generates a magnetic field around the material. The field is in a far distance, that of the magnetic moment of $LS|\boldsymbol{I}| = V|\boldsymbol{M}|$, where S is the area of ends of the material, L is the length. Thereofre, measurement of outer magnetic field originated from the material can give $\boldsymbol{M}$. In the discussion

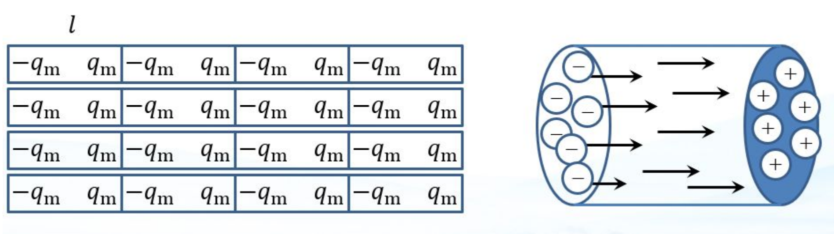


Fig. 1.4 A naive model of magnetization of a material which is composed of many small magnets with length l and magnetic charges $\pm q_{\rm m}$.

of magnetism, i.e., magnetic properties of materials, (apart from how far the above simple model can be used) what we should consider in the first place is **how magnetic moments are induced in the material ?**

1.1.4 Electromagnetic field in the presence of materials

Since the magnetic moment is expressed as a circular current, the effect of the magnetic moment in a substance can also be expressed as a current. As illustrated in Fig. 1.5, we take the coordinate $\boldsymbol{r}'$ inside the material and sum up the effect of local moments to give

$$\begin{aligned}\boldsymbol{A} &= \frac{\mu_0}{4\pi}\int_{\rm mat} dv' \frac{\boldsymbol{M}'\times\boldsymbol{r}}{r^3} = -\frac{\mu_0}{4\pi}\int_{\rm mat} dv' \left(\boldsymbol{M}'\times\nabla\frac{1}{r}\right) \\ &= \frac{\mu_0}{4\pi}\int_{\rm mat} dv' \left(\boldsymbol{M}'\times\nabla'\frac{1}{r}\right). \end{aligned} \tag{1.11}$$

Here the integration volume is taken inside the material. The symbols with prime as $\boldsymbol{M}'$ means they are expressed as a function of the coordinate (x', y', z'). Further partial integration gives

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_{\rm mat} dv' \frac{\nabla'\times\boldsymbol{M}'}{r}. \tag{1.12}$$

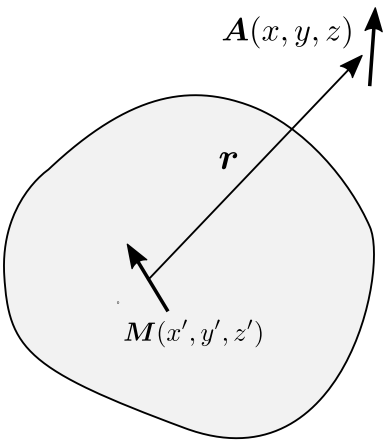


Fig. 1.5 Illustration of vector potential formed by magnetic moments inside a material

Then if we write

$$\boldsymbol{j_M} \equiv \nabla\times\boldsymbol{M}, \tag{1.13}$$

from $\nabla\cdot\boldsymbol{j_M} = 0$, we can view $\boldsymbol{j_M}$ as a kind of electric current.

We add the true current of real charge $\boldsymbol{j}$ to the above "equivalent current" to obtain the vector potentioal as

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int dv' \frac{\boldsymbol{j}' + \boldsymbol{j}'_{\boldsymbol{M}}}{r}. \tag{1.14}$$

Then if we write

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{\mu_0}{4\pi}\int dv' \frac{(\boldsymbol{j}' + \boldsymbol{j}'_{\boldsymbol{M}})\times\boldsymbol{r}}{r^3}, \tag{1.15}$$

we obtain the relation

$$\nabla\times\boldsymbol{B} = \mu_0(\boldsymbol{j}+\boldsymbol{j_M}) = \mu_0\boldsymbol{j} + \mu_0\nabla\times\boldsymbol{M}. \tag{1.16}$$

Defining magnetic field as

$$\boldsymbol{H} \equiv \boldsymbol{B}/\mu_0 - \boldsymbol{M}, \tag{1.17}$$

we find

$$\nabla\times\boldsymbol{H} = \boldsymbol{j}, \tag{1.18}$$

namely $\boldsymbol{H}$ does not depend on the equivalent current of magnetic moment.

Now we consider the electromagnetic field in the presence of a material. Electric flux density $\boldsymbol{D}$, and magnetic flux density $\boldsymbol{B}$ is given by

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} \tag{1.19a}$$
$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}), \tag{1.19b}$$

where $\boldsymbol{P}$ is electric polarization. In the case of electric charge, true charge $\rho_{\rm t}$ exists other than the polarization charge $\rho_{\rm p}$. From $\rho_{\rm p} = -\nabla\cdot\boldsymbol{P}$ we can write

$$\nabla\cdot\boldsymbol{D} = \rho_{\rm t}, \tag{1.20}$$

which means the effect of polarization in taken into $\boldsymbol{D}$. Further, if we express $\nabla\times\boldsymbol{H}$ with $\boldsymbol{D}$, we obtain the formula

$$\nabla\times\boldsymbol{H} = \boldsymbol{j} + \frac{\partial\boldsymbol{D}}{\partial t}, \tag{1.21}$$

which is in the same form as (1.1d) and the effect of magnetization is included into $\boldsymbol{H}$.

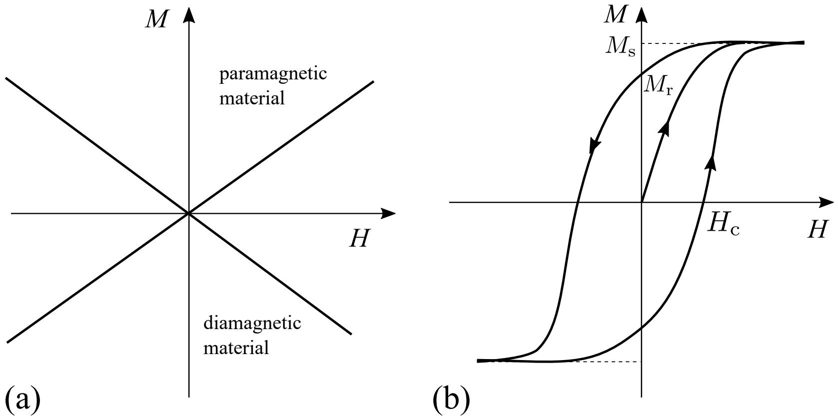


Fig. 1.6 (a) Schematic diagrams of M-H curves in paramagnetic ad diamagnetic materials. (b) Schematic diagram of M-H curve in ferromagnetic material. H_c: coercive force, M_r: remanent magnetization, M_s: saturation magnetization.

1.1.5 M(B)-H curve

As above, the response of materials to magnetc field $\boldsymbol{H}$ appears as magnetization $\boldsymbol{M}$. When $\boldsymbol{M}$ is proportional to $\boldsymbol{H}$, we write

$$\boldsymbol{M} = \chi \boldsymbol{H}, \tag{1.22}$$

where the coefficient χ is called **magnetic susceptibility**. Then we write

$$\boldsymbol{B} = (\chi + \mu_0)\boldsymbol{H} \equiv \mu^* \boldsymbol{H}, \tag{1.23}$$

and call $\mu^* \equiv \chi + \mu_0$ **magnetic permeability**. $\bar{\mu} \equiv \mu^*/\mu_0 = \bar{\chi} + 1 = \chi/\mu_0 + 1$ is called **relative permiability**.

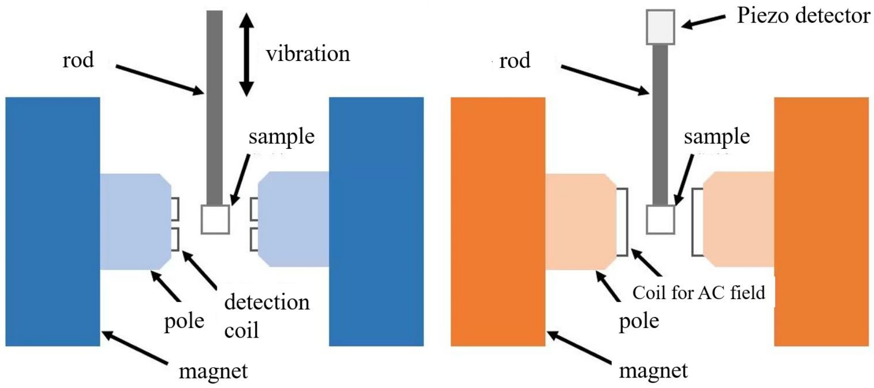
Among such materials with linear response, we call **paramagnetic** material for those with $\chi > 0$ and **diamagnetic** material for those with $\chi < 0$. And there are many materials such as **ferromagnetic** materials in which the linear relation in eq. (1.22) does not hold. These materials are very important particularly in the field of magnetics – field of application. In such cases, behavior of magnetization is often presented in the form of **M-H curve**, in which magnetization M is plotted versus H. Figure 1.6 shows a schematic of M-H curve. In the case of linear response in eq. (1.22), the M-H curve should be linear as shown in Fig. 1.6(a). On the other hand, Fig. 1.6(b) shows a schematic of ferromagnetic response. With up-down sweeps of H, the response of M is strongly non-linear, and different behavior is observed for the direction of the sweeps, which phenomenon is called **hysteresis**. B-H curves are also adopted for giving the same information.

1.2 Measurement of magnetization

Here we introduce some of experimental methods for the measurement of magnetization before going into the theories. Also we touch on the problem of demagnetizing field, which is important for measurement.

1.2.1 Methods of magnetization measurement

Below we list some representative methods. There are two major ways for the measurement: (a) Measurement of magnetic field caused by magnetization; (b) Variation in magnetization is detected as voltage caused by electromagnetic induction. Examples of (a) are, vibrating sample magnetometer (VSM), superconducting quantum interference device (SQUID) magnetometer. An example of (b) is a pick-up coil type magnetometer.



(a) (b)

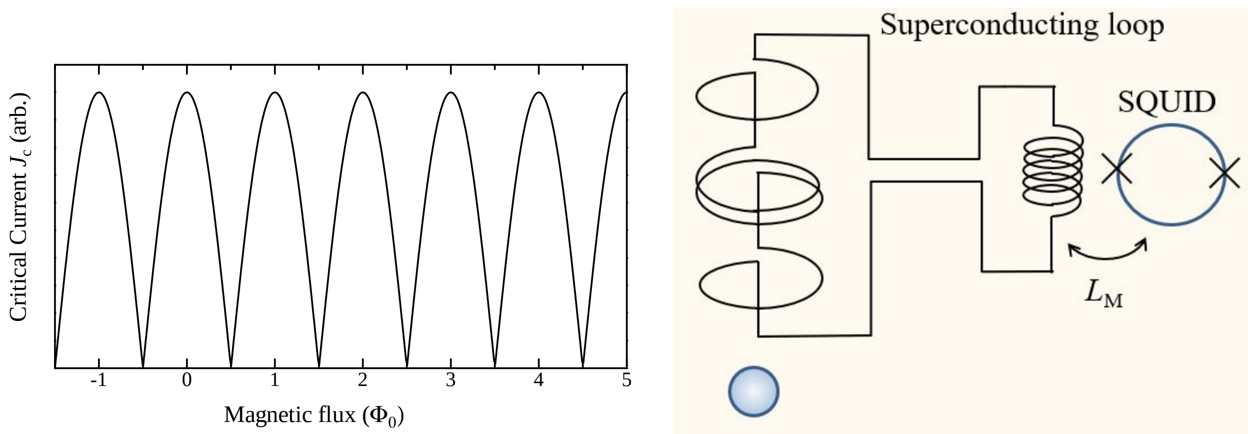
Fig. 1.7 (a) Principle of magnetization measurement in VSM. (b) Principle of magnetization measurement in AGM. From https://www.toyo.co.jp/material/casestudy/detail/id=7003

(a1) **Vibrating sample magnetometer**: Figure 1.7(a) shows the principle of magnetization measurement by VSM. When a magnetized substance is spatially vibrated, the magnetic field generated by the magnetization is not spatially uniform, so it vibrates with time when viewed at a fixed point in space. Then in a detection coil placed at the point the vibrating field procuces an alternating voltage though electromagnetic induction. In VSM the ditection of the AC voltage gives measurement of magnetization.

In actual use of VSM, as annotated in the figure, the sample, in addition to the vibration, moves slowly through two inversely-wound detection coils to produce a signal peak and dip, which are mirrored to each other. With this devicing, the signal offset can be eliminated.

(a2) **Alternating-gradient magnetometer**: AGM[1] utilizes the fact that a magnetic moment gets force in magnetic field gradient. As shown in Fig. 1.7(b), alternative current in coils attached to an external magnet provide a vibration in field gradient. Then the vibration in the force on the suspender is detected to give the magnetization. Force detection has become extremely sensitive by the method using a laser and a cantilever, which is familiar with atomic force microscopes (AFM) (the piezo element is used in the figure), so extremely high sensitivity can be obtained.

(a3) **SQUID magnetometer**: Magnetic flux piercing a superconducting ring is quantized by the unit of quantum flux $\Phi_0 \equiv h/2e \approx 2.07 \times 10^{-15}$ Wb. A superconducting quantum interference device (SQUID) is a superconducting ring



(a) (b)

Fig. 1.8 (a) Schematic diagram of critical current in a SQUID device as a function of magnetic field. (b) Superconducting circuit diagram of a SQUID magnetometer.

with some (in the case of dc-SQUID, two) weak links. The superconducting critical current J_c through it largely oscillates with magnetic flux with the period of Φ_0 as illustrated in Fig. 1.8(a). The unit Φ_0 itself is already very small and the output of a SQUID is strongly non-linear as shown in Fig. 1.8(a), which fact enables us to measure orders of smaller change in the magnetic flux.

Because SQUID sensors should be placed in weak magnetic fields, a closed loop of superconductor is used as shown in Fig. 1.8(b). One end of the loop is magnetically coupled to a SQUID sensor through a coil. As in VSMs, a specimen moves in counter-wound superconducting coil (in the figure the winding is $-++-$), which form the closed loop. The function fitting to the lineshape gives precise value of magnetization. To get high S/N, the specimen should be suspended by a uniform substance with small magnetic susceptibility. The upper limit of the field is determined by superconducting critical field of the pick-up coil. Because the SQUID magnetometers have very high sensitivity, they are applied for measurement of environmental magnetic field or magnetic field leakage from brains (magnetoencephalograph).

(a4) **NV center magnetometer**: Lattice defects formed as complex of a nitrogen (N) impurity and a vacancy (V) in diamonds have quantum states in the bandgap. Combination of optical excitation/detection and electron spin resonance with microwave provides highly sensitive detection of very local magnetic field. S/N can be highly enhanced by, e.g., combination of spin-rotation pulse sequences. In some cases, the sensitivity is as high as the replacement of SQUID magnetometers.

(b1) **Pick-up coil method**: It is used that the response of the coil changes depending on the magnetic permeability inside the coil. A specimen is inserted into a coil. When an AC magnetic field is appled, a voltage is induced on the coil due to the flux variation. Integration of the signal gives the variation of magnetic flux and thus the magnetization.

1.2.2 Effect of demagnetizing field

One thing to note when measuring magnetization is the effect of demagnetizing field. This appears as shape dependence, especially when measuring the magnetization of ferromagnets. In the magnetic charge model, when a material with a finite size is placed in a magnetic field and magnetized, magnetic charges (magnetic poles) appear at both ends of the sample, which creates a magnetic field inside the sample in the opposite direction to the external field. This is the **demagnetizing field** (Fig. 1.9(a)). The demagnetizing field H_d is proportional to the magnetization M as

$$H_\mathrm{d} = N\frac{M}{\mu_0}, \tag{1.24}$$

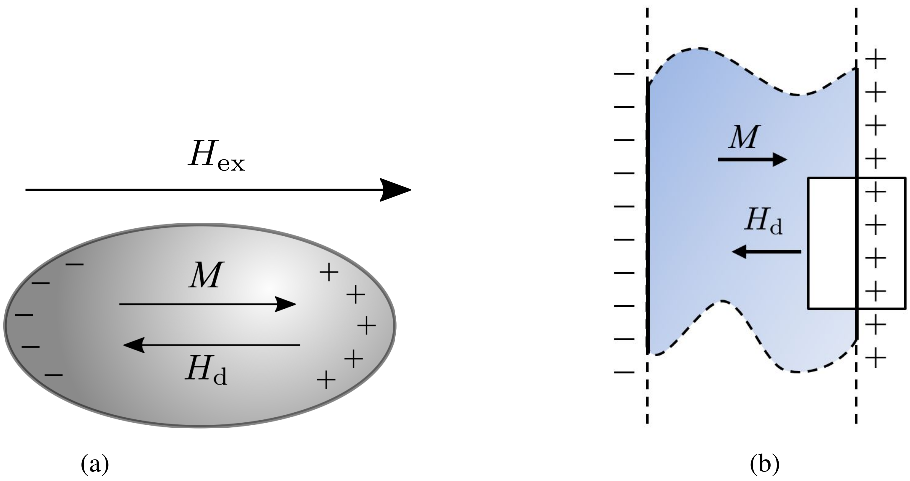


Fig. 1.9 (a) Conceptual diagram of demagnetic field. (b) Calculation of demagnetic field coefficient of plate-shaped sample.

where N is called **demagnetizing factor**. The demagnetizing factor depends on the shape of specimen. For needle-like thin specimen, it is almost zero while large for thick and short ones.

As the simplest case, we consider the case of plate-like sample in Fig. 1.9(b). For magnetization M, the areal densities of surface magnetic charge is $\pm M$. From the symmetry, the magnetic force lines are perpendicular to the plane. Applying the Gauss theorem to a cylinder containing both front and back of the plate, we know that no magnetic flux comes out because the total magnetic charge in the cylinder is zero. Here we again apply the Gauss theorem to another cylinder which contains just one of the surface with a bottom of unit area to obtain

$$\int_{\rm surface} H_{\rm n} ds = H_{\rm d} = \frac{M}{\mu_0},$$

which gives the demagnetizing factor $N = 1$. Demagnetizing factors for various shapes have been calculated.

When there is large demagnetizing field, e.g., hysteretic M-H curve as in Fig. 1.6(b) is largely distorted. The following example is introduced in ref. [2]. Permalloy (an alloy of Fe and Ni, symbol Py) has a very small coercive force 2 A/m ($\approx$ 0.025 Oe), and under usual condition, the magnetization saturates at very small external fields. However, if we make a sphere with Py, becase the saturation field of Py is about 9.23×10^5 A/m and the demagnetizing factor of sphere is 1/3, the demagnetizing field amounts to 3.08×10^5 A/m (=3860 Oe). That is, saturation magnetization cannot be obtained unless a magnetic field of about 100,000 times that without a demagnetizing field is applied. Normally, when submitting data as an M-H curve to a scientific paper, etc., it is necessary to correct the demagnetizing field or to state that it has not been corrected.

1.3 Classical theory of magnetization

Let us go into the physical mechanism of magnetic properties. Though magnetism still has many unsolved problems the present understandings have been obtained with quantum mechanics. Within the classical theory, even elementary understanding is difficult. We see that in this section. Here we refer to "classical theory" that inside materials exists a group of electrons which are classical particles with charge $-e$ and mass m. And they create the equivalent current (1.13).

1.3.1 Classical treatment of paramagnetic moment

Before digging into how to deal with magnetism in classical mechanics and statistics, let us assume that there is already a magnetic moment in matter that has a degree of freedom to change direction, and see the consequence. Matter is composed of atoms, and we will consider a model in which the electrons around the nucleus originally have a magnetic moment due to orbital motion.

Let us consider a set of molecules in a magnetic field along z-axis with flux density $\boldsymbol{B}$. Each molecule has independent magnetic moment μ. The magnetic energy of a moment is $U = -\boldsymbol{\mu}\cdot\boldsymbol{B} = -\mu B\cos\theta$. In classical statistics, the average of z-component in the moment over the is given by

$$\begin{aligned}
\langle\mu_z|\mu_z\rangle &= \int \exp\left(-\frac{U}{k_{\rm B}T}\right)\mu_z d\Omega\Big/\int \exp\left(-\frac{U}{k_{\rm B}T}\right)d\Omega \\
&= \int \exp\left(\frac{\mu B\cos\theta}{k_{\rm B}T}\right)\mu\cos\theta d\Omega\Big/\int \exp\left(\frac{\mu B\cos\theta}{k_{\rm B}T}\right)d\Omega \\
&= k_{\rm B}T\frac{\partial}{\partial B}\log\left[2\pi\int_0^\pi \exp\left(\frac{\mu B\cos\theta}{k_{\rm B}T}\right)\sin\theta d\theta\right] \\
&= \mu\left[\coth\left(\frac{\mu B}{k_{\rm B}T}\right) - \frac{k_{\rm B}T}{\mu B}\right],
\end{aligned} \tag{1.25}$$

where Ω is the solid angle. In high temperature approximation $\mu B \ll k_{\rm B}T$, the average is

$$\frac{\langle\mu_z|\mu_z\rangle}{B} \sim \frac{\mu^2}{3k_{\rm B}}\frac{1}{T}. \tag{1.26}$$

This indicates **Curie law**, that is the magnetic susceptibility is inversely proportional to temperature.

1.3.2 Classical theory of diamagnetism

We consider an electron moving along a circle with radius r in xy plane. We write the circle as Γ and the area as S, and write down the integral form of the Maxwell equation (1.1b) as

$$\oint_\Gamma \boldsymbol{E}\cdot d\boldsymbol{l} = -\frac{\partial}{\partial t}\int_{\rm S}\boldsymbol{B}\cdot d\boldsymbol{\sigma}. \tag{1.27}$$

From this, the induced electromotive force for variation of magnetic flux B is given as

$$2\pi rE = -\frac{\partial}{\partial t}\left(B\pi r^2\right) \quad \therefore E = -\frac{r}{2}\frac{dB}{dt}. \tag{1.28}$$

The electron is accelerated by $-eE$ to the tangential direction and the time derivative of angular momentum L is

$$\frac{dL}{dt} = r\times(-eE) = e\frac{r^2}{2}\frac{dB}{dt}. \tag{1.29}$$

Then the shift $0\to B$ creates the angular momentum $L = e\frac{r^2}{2}B$. The velocity of electron also increases from 0 to v then $v = L/mr$. As describe in Appendix 1B.2, the magnetic moment of this circular current is (the area of circle)×(crrent). Then it is

$$\mu = SJ = \pi r^2\frac{ev}{2\pi r} = \pi r^2\frac{L}{mr}\frac{e}{\pi r} = \frac{e}{2m}e\frac{r^2}{2}B. \tag{1.30}$$

If we replace r, the distance between nucleous and electron, with the average $\left\langle x^2+y^2|x^2+y^2\right\rangle_{\rm av}$,

$$\mu = -\frac{e^2}{4m}\left\langle x^2+y^2|x^2+y^2\right\rangle_{\rm av}B. \tag{1.31}$$

1.3.3 Breakdown of classical theory of magnetism

In the above, seemingly reasonable results have been obtained even within classical mechanics. However, below, we see the theory is actually broken down in a simple discussion. We introduce electromagnetic field to simple single-particle hamiltonian $\mathcal{H} = \boldsymbol{p}^2/2m$ by changing energy $\mathcal{E}$ and momentu $\boldsymbol{p}$ as

$$\mathcal{E}\to\mathcal{E}+e\phi, \quad \boldsymbol{p}\to\boldsymbol{p}+e\boldsymbol{A}. \tag{1.32}$$

That is the hamiltonian is written as

$$\mathcal{H} = \frac{1}{2m}(\boldsymbol{p}+e\boldsymbol{A})^2 - e\phi. \tag{1.33}$$

This way of introduction is justified by the derivation of canonical equation

$$m\frac{d\boldsymbol{v}}{dt} = -e[\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}],$$

which reproduces the Lorentz force.

We take symmetric gauge $\boldsymbol{A} = (\boldsymbol{B}\times\boldsymbol{r})/2$. Then (1.33) is calculated as follows.

$$\mathcal{H} = \frac{\boldsymbol{p}^2}{2m} + \frac{e}{2m}(\boldsymbol{r}\times\boldsymbol{p})\cdot\boldsymbol{B} + \frac{e^2}{8m}(\boldsymbol{B}\times\boldsymbol{r})^2. \tag{1.34}$$

From this, the magnetic dipole moment $\mu_{\rm m}$ originates from the motion of electrom is calculated as

$$\mu_{\rm m} = -\frac{\partial \mathcal{H}}{\partial B} = -\frac{e}{2m}(\boldsymbol{r}\times\boldsymbol{p}) - \frac{e^2}{4m}(\boldsymbol{r}\times(\boldsymbol{B}\times\boldsymbol{r})). \tag{1.35}$$

Here, the first term in rhs is called **paramagnetic term**, which is proportional to the angular momentum $\boldsymbol{r}\times\boldsymbol{p}$. On the other hand, the second term is proportional to magnetic flux density $|\boldsymbol{B}|$, which indicates that the inductive electric field accelerates the electron and that this comes form the eddy current-like motion to cancel the external field. Hence this is called **diamagnetic term**.

Let us move on to an N-particle electron system. The hamiltonian is written with writing electron-electron interaction as V as

$$\mathcal{H}_N = \sum_{n=1}^{N}\left[\frac{1}{2m}\left(\boldsymbol{p}_n + e\boldsymbol{A}(\boldsymbol{r}_n)\right)^2 - e\phi(\boldsymbol{r}_n)\right] + V(\boldsymbol{r}_1,\boldsymbol{r}_2,\cdots,\boldsymbol{r}_N). \tag{1.36}$$

The partition function Z at temperature T is

$$Z = \prod_{n=1}^{N}\int \frac{d\boldsymbol{r}_n d\boldsymbol{p}_n}{h^3} e^{-\mathcal{H}/k_{\rm B}T}. \tag{1.37}$$

Though it is classical, it has the Planck constant h because we need to calculate the number of states and that should be based on a unit space in the $\boldsymbol{r}$-$\boldsymbol{p}$ phase space.

Here we write $\boldsymbol{\pi}_n = \boldsymbol{p}_n + e\boldsymbol{A}(\boldsymbol{r})$, then

$$\begin{aligned} Z &= \prod_{n=1}^{N}\int \frac{d\boldsymbol{r}_n d\boldsymbol{\pi}_n}{h^3} e^{-\mathcal{H}'/k_{\rm B}T}, \\ \mathcal{H}' &= \sum_{n=1}^{N}\left[\frac{\boldsymbol{\pi}_n^2}{2m} - e\phi(\boldsymbol{r}_n)\right] + V(\boldsymbol{r}_1,\boldsymbol{r}_2,\cdots,\boldsymbol{r}_N), \end{aligned} \tag{1.38}$$

which has no $\boldsymbol{A}$ in the expression. Hence the statistical average of the magnetic moment is naturally

$$\langle\boldsymbol{\mu}_{\rm m}|\boldsymbol{\mu}_{\rm m}\rangle = -\frac{1}{N}\frac{\partial F}{\partial \boldsymbol{B}} = \frac{1}{Nk_{\rm B}T}\frac{\partial \ln Z}{\partial \boldsymbol{B}} = \langle\boldsymbol{\mu}_{\rm para}|\boldsymbol{\mu}_{\rm para}\rangle + \langle\boldsymbol{\mu}_{\rm dia}|\boldsymbol{\mu}_{\rm dia}\rangle = 0. \tag{1.39}$$

That is, within the classical picture of electrons and classical statistics, the pramagnetic and diamagnetic terms cancel each other and there shuold be no magnetism in this system. This is called **Bohr-van Leeuwen theorem**.

1.4 Spin and magnetic moment of electron

There are several factors that cause magnetism in quantum theory, one of which is that in quantum theory, an electron has a spin and a spin magnetic moment. The question why a point charge like an electron can have an internal freedom called spin and that also has a magnetic moment can be answered clearly by relativistic quantum mechanics. All of you should have already learned about this in the undergraduate course. However we would like review here the beautiful logics how this is derived[3].

1.4.1 Dirac equation

We consider quantum mechanics in the form of wavefunction and try to find out a form which stands with the special relativity. Schrödinger equation is a non-relativistic approximation and for the above purpose, we need to find out the form which is invariant for Lorentz transformations.

The one-dimensional Schrödinger equation is obtained by replacing the energy and the momentum as

$$E \to i\hbar\frac{\partial}{\partial t}, \quad p \to -i\hbar\frac{\partial}{\partial x}, \tag{1.40}$$

in the energy-momentum relation $E = p^2/2m$ in Newtonian mechanics. Furthermore, the wavefunction in the wave mechanics should have the meaning of probability amplitude and hence the differential equation for the wavefunction should be that of single-derivtive in time[4].

First the relativistic energy-momentum relation is

$$E^2 = (pc)^2 + (mc^2)^2. \tag{1.41}$$

If the transformation (1.40) is applied directly to the above we obtain Klein-Gordon equation but this is in the second order in time and does not fulfil the condition for the wavefunction. If the differential equation is first order in time and the orders for time and space should be the same, the derivative on the space also should be the first order. Then we write

$$E = \sum_{k=1,2,3} \alpha_k p_k c + \beta mc^2, \tag{1.42}$$

and try to compromise the above with eq. (1.41). Taking the square of the lhs and the equation to be eq. (1.41), the conditions are

$$\begin{cases} \alpha_k^2 = 1, \quad \beta^2 = 1, & (1.43a) \\ \alpha_k\alpha_j + \alpha_j\alpha_k = 0 \ (k \neq j), & (1.43b) \\ \alpha_k\beta + \beta\alpha_k = 0. & (1.43c) \end{cases}$$

In order to satisfy the above, we consider matrices for α_k, β and the dimension should be at least 4×4.

Then the wavefunction should have four components. The equation for the 4-component wavefunction $\boldsymbol{\psi} = {}^t(\psi_1, \psi_2, \psi_3, \psi_4)$ (^{t} means transpose) should be

$$i\hbar\frac{\partial\boldsymbol{\psi}}{\partial t} = \left[-i\hbar c \sum_{k=x,y,z} \alpha_k \frac{\partial}{\partial x_k} + \beta mc^2\right]\boldsymbol{\psi} \tag{1.44a}$$

$$\equiv \mathcal{H}_{\rm D}\boldsymbol{\psi}, \quad \mathcal{H}_{\rm D} = c\boldsymbol{\alpha}\boldsymbol{p} + mc^2\beta. \tag{1.44b}$$

To obtain a specific form of α_k, β, we introduce the following **Pauli matrices**:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.45}$$

These matrices have the following relations

$$\sigma_i\sigma_j = -\sigma_j\sigma_i = i\sigma_k, \quad \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I. \tag{1.46}$$

Here for (i, j, k), (x, y, z) are assigned with cyclic rotations. From the above we reach a specific representation (Pauli representation) as follows.

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{1.47}$$

Pauli representaion is one of the possible representations and we can find infinite numbers of representation with unitary transformation. Calculated results for the observables should be the same for all the representations. The four dimensions correspond to the spin degree of freedom and the freedom of particle-antiparticle (isospin). These four freedoms exist on the spatially point charge of electron-positron. In conclustion, this is the equation for the particle with spin 1/2 and with a finite mass.

1.4.2 Spin angular momentum

We consider a centeral force potential $V(\boldsymbol{r})$ and write

$$\mathcal{H} = \mathcal{H}_{\rm D} + V(\boldsymbol{r}). \tag{1.48}$$

The angular momentum

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \tag{1.49}$$

does not commute with the hamiltonian (1.48) as

$$[\boldsymbol{L}, \mathcal{H}] = i\boldsymbol{\alpha} \times \boldsymbol{p}. \tag{1.50}$$

$\boldsymbol{\alpha}$ is a vector of components α_k. We expand Pauli matrices to 4×4 dimension as

$$\sigma_k^{(4)} = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix},$$

and write the vector of elements $\sigma_{x,y,z}^{(4)}$ as $\boldsymbol{\sigma}$. Then from the relation

$$[\boldsymbol{\sigma}, \mathcal{H}] = -2i\boldsymbol{\alpha} \times \boldsymbol{p}/\hbar, \tag{1.51}$$

we define the total angular momentum $\boldsymbol{J}$ as

Total angular momentum

$$\boldsymbol{J} = \boldsymbol{L} + \frac{\hbar}{2}\boldsymbol{\sigma} \equiv \boldsymbol{L} + \boldsymbol{s}, \tag{1.52}$$

then we obtain

$$[\boldsymbol{J}, \mathcal{H}] = 0, \tag{1.53}$$

that means $\boldsymbol{J}$ is a constant of motion. Namely $\boldsymbol{s} \equiv (\hbar/2)\boldsymbol{\sigma}$ is an observable which has characteristics of angular momentum. This is the **spin angular momentum**. We reach the conclusion that though an electron is a point in the space, it has an angular momentum as if it has a rotation.

1.4.3 Magnetic moment

We consider the Dirac equation in the presence of static electromagnetic fields. Just like eq. (1.32), we introduce a scalar and a vector potential to obtain

$$i\hbar\frac{\partial\psi}{\partial t} = [c\boldsymbol{\alpha}(\boldsymbol{p} + e\boldsymbol{A}) + \beta m - e\phi]\,\psi. \tag{1.54}$$

Now we rewrite it to

$$\left[\left(i\hbar\frac{\partial}{\partial t} + e\phi\right) - c\sum_{j=x,y,z}\alpha_j\left(-i\hbar\frac{\partial}{\partial r_j} + eA_j\right) - \beta mc^2\right]\psi = 0. \tag{1.55}$$

Then operate

$$i\hbar\frac{\partial}{\partial t} + e\phi + c\sum_{j=x,y,z}\alpha_j\left(-i\hbar\frac{\partial}{\partial r_j} + eA_j\right) + \beta mc^2 \tag{1.56}$$

from the left. After some algebra by using the commutation relations of α_j, β, we reach

$$\left[\left(i\hbar\frac{\partial}{\partial t} + e\phi\right)^2 - c^2(\boldsymbol{p} + e\boldsymbol{A})^2 - m^2c^4 + ic\hbar e(\boldsymbol{\alpha}\cdot\boldsymbol{E}) + i\hbar c^2 e(\alpha_x\alpha_y B_z + \alpha_y\alpha_z B_x + \alpha_z\alpha_x B_y)\right]\psi = 0. \tag{1.57}$$

Here from

$$\alpha_x\alpha_y = i\sigma_z^{(4)}, \quad \alpha_y\alpha_z = i\sigma_x^{(4)}, \quad \alpha_z\alpha_x = i\sigma_y^{(4)} \tag{1.58}$$

we can write (1.57) as

$$\left[\left(i\hbar\frac{\partial}{\partial t} + e\phi\right)^2 - c^2(\boldsymbol{p} + e\boldsymbol{A})^2 - m^2c^4 + ic\hbar e(\boldsymbol{\alpha}\cdot\boldsymbol{E}) - \hbar c^2 e\boldsymbol{\sigma}\cdot\boldsymbol{B}\right]\psi = 0. \tag{1.59}$$

To obtain steady state we write

$$\psi(\boldsymbol{r},t)=\exp(-i\epsilon t/\hbar)\varphi(\boldsymbol{r}) \tag{1.60}$$

and obtain

$$\left[(\epsilon+e\phi)^2-c^2(\boldsymbol{p}+e\boldsymbol{A})^2-m^2c^4+ic\hbar e(\boldsymbol{\alpha}\cdot\boldsymbol{E})-\hbar c^2 e\boldsymbol{\sigma}\cdot\boldsymbol{B}\right]\varphi=0. \tag{1.61}$$

Here we put $\phi=0$, $\boldsymbol{E}=0$. In non-relativistic approximation $p\ll mc$, i.e., $\epsilon\approx mc^2$, hence we write $\epsilon=mc^2+\delta$ and ignore the power of δ/mc^2 higher than the second order to obtain[*2]

$$\left[\frac{1}{2m}(\boldsymbol{p}+e\boldsymbol{A})^2+\frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}\right]\varphi=\delta\varphi. \tag{1.62}$$

Here we define the quantity called **Bohr magneton** as

Bohr magneton

$$\mu_{\rm B}\equiv\frac{e\hbar}{2m}\approx 9.274\times10^{-24}\ \mathrm{JT}^{-1}, \tag{1.63}$$

then in eq. (1.62) the term related to the magnetic field is

$$\frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}=\mu_{\rm B}\boldsymbol{\sigma}\cdot\boldsymbol{B}=\frac{2}{\hbar}\mu_{\rm B}\boldsymbol{s}\cdot\boldsymbol{B}. \tag{1.64}$$

This means that an electron has, with the spin angular momentum $\boldsymbol{s}$, a magnetic moment $-2\mu_{\rm B}\boldsymbol{s}/\hbar$

Appendix 1A Unit systems of electromagnetism

The unit system, or metrology is extremely important for human life even beyond the boundaries of science or the framework of scholarship. In usual systems, at first place a small number of basic units are determined by some way, and then other units are determined through universal laws of physics. Naturally, there are many possible ways to constitute unit systems. We should choose a unit system convenient for the problem that we need to tackle from many possible ones, particularly in the case of electromagnetism. Many researchers, even among non-expert in unit, have their own opinions for unit systems. Hence it is impossible to force everyone to choose one. In spite of such tendency, the SI (Système International) unit system is defined as an international standard in consideration of practicality, logical consistency, and historical continuity. I would like to adopt the SI unit system as far as I can, but here, I introduce major systems of unit in electromagnetism in very short. If you are interested in the metrology, I would like to recommend Ref. [5]. Care should be taken, though, that the committee meeing of the SI unit is held annually, and the unit may have big changes as did in 2019. It is necessary to refer to the web etc. for the latest definition. Here I summarize CGS-esu (electrostatic system of unit) and MKSA (SI) system of unit very briefly.

1A.1 CGS-esu

The Coulomb's law should be written in the form

$$F=k_{\rm q}\frac{q_1q_2}{r^2}. \tag{1A.1}$$

In CGS electrostaticunit, $k_{\rm q}$ is just 1 and with no physical dimension. Then the electric charge can be expressed by [L], [M], [T] as

$$[Q]=[M^{1/2}L^{3/2}T^{-1}].$$

[*2] This φ has four components and then we need to reduce the dimension to two with the non-relativistic approximation. But here we skip the procedure for simplicity. The result here is the same for this but to derive spin-orbit interaction, this is indispensable.

When the force between electric charges with the same amount at the distance of 1 cm is 1 dyn, the charge is 1 esu (CGS esu). There are three basic quantities and there is no factor of $(4\pi)^{-1}$ in the Coulomb's law (this means 4π appears in the Maxwell equation). Hence this unit system is classified to three components irrational unit system.

1A.2 MKSA unit system

Until a while ago, the current was introduced as the fourth basic quantity, the unit of the current was A (ampere), and 1 A was defined from the force acting between the parallel conductors separated by 1 m in a vacuum with a length of 1 m as to be 2×10^{-7} N. However in the redefinition in 2019, the fourth basic quantity becomes the charge, which is introduced by determining the elementary charge e as $1.602176634\times10^{-19}$ C (coulomb). Then the current is introduce by the charge and the time (A·s =C). Because the number of basic quantities is four and $k_{\rm q}$ has the factor $(4\pi)^{-1}$, the unit system is classified to four components rational unit system.

Appendix 1B: Dipole field and dipole inteaction

1B.1 Magnetic field created by magnetic dipole

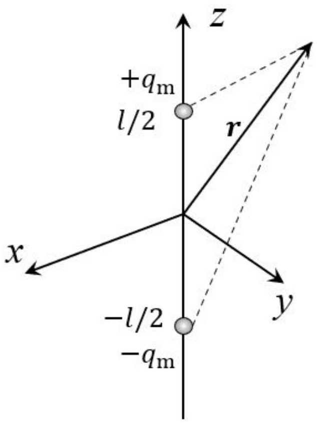


Fig. 1B.1 Magnetic charge model of magnetic dipole.

From eq. (1.1d), when there is no true current and no time-derivative of electric field, the rotation of magnetic field $\nabla\times\boldsymbol{B}$ is zero. In such a case, we can define magnetostatic potential $\phi_{\rm m}$ as a scalar function of spatial coordinage $\boldsymbol{r}$. The magnetic field is given by

$$\boldsymbol{H} = -\nabla\phi_{\rm m}. \tag{1B.1}$$

We consider the magnetostatic potential $\phi_{\rm m}$ for the situation that magnetic charges of $\pm q_{\rm m}$ are placed at $\pm\boldsymbol{p} = (0, 0, \pm l/2)$ respectively.

$$\phi_{\rm m}(\boldsymbol{r}) = \frac{1}{4\pi\mu_0}\left(\frac{q_{\rm m}}{|\boldsymbol{r}-\boldsymbol{p}|} - \frac{q_{\rm m}}{|\boldsymbol{r}+\boldsymbol{p}|}\right). \tag{1B.2}$$

We assume that l is sufficiently smaller than $|\boldsymbol{r}|$ ($|\boldsymbol{r}| \gg l$), and expand $\phi_{\rm m}$ with the power of l and take the first order term to obtain

$$\phi_{\rm m}(\boldsymbol{r}) = \frac{q_{\rm m}}{4\pi\mu_0}\frac{lz}{r^3} = \frac{q_{\rm m}l}{4\pi\mu_0}\frac{\cos\theta}{r^2}. \tag{1B.3}$$

The last expression is on the polar coordinate (r, θ, φ).

We write $\boldsymbol{l} = (0, 0, l)$, and the magnetic moment as $\boldsymbol{\mu} = q_{\rm m}\boldsymbol{l}/\mu_0$, the magnetic field originates from the magnetic dipole is

$$\boldsymbol{B} = -\frac{1}{4\pi\mu_0}\nabla\left(\frac{\boldsymbol{\mu}\cdot\boldsymbol{r}}{r^3}\right). \tag{1B.4}$$

We apply the polar coordinate representation of ∇

$$\nabla = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi \tag{1B.5}$$

to eq. (1B.3), and we reach

$$B_r = \frac{|\boldsymbol{\mu}|}{4\pi\mu_0}\frac{2\cos\theta}{r^3}, \quad B_\theta = \frac{|\boldsymbol{\mu}|}{4\pi\mu_0}\frac{\sin\theta}{r^3}, \tag{1B.6}$$

which is nothing but eq. (1.6).

1B.2 Dipole field as shrink limit of circular current

In the text, we considered the fictitious "magnetic charge" to make the physical picture simpler. The experiments so far have shown that there is no isolated magnetic monopole like electric monopole. Then there is an opinion that we should not use the magnetic charge even as a mathematical concept. On the other hand, there are researchers who positively use the magnetic charge as an established physical concept [2], because there are many such physically established concepts like the vector potential which is not an observable but a mathematical method.

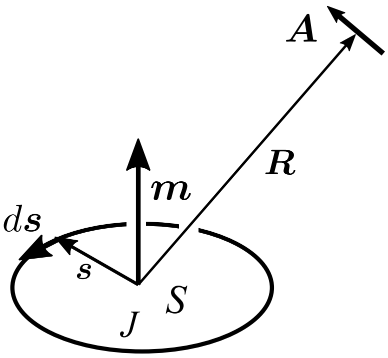


Fig. 1B.2 Definition of magnetic dipole by circular current.

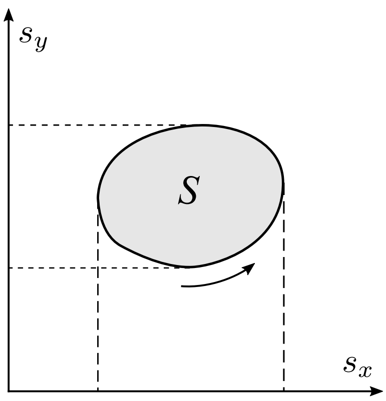
This problem also relates to the view of electric-magnetic symmetry in the Maxwell equation (1.1). That is, so called the problem whether $\boldsymbol{E}$-$\boldsymbol{B}$ formulation or $\boldsymbol{E}$-$\boldsymbol{H}$ formulation. In the former, the magnetic field is introduced along the Viot-Savart law as formed by a current on a wire element while in the latter the field is formed by magnetic charges through the Coulomb law. In CGS unit system, these two ways of introduction do not give significant difference. However in MKSA unit system, this leads to the difference in the unit of magnetization. The same problem appears when we introduce magnetic dipole through magnetic charges. When we consider a pair of moment (1.2), the selection of $\boldsymbol{E}$-$\boldsymbol{H}$ correspondence or $\boldsymbol{E}$-$\boldsymbol{B}$ correspondence results in the difference whether eq. (1.3) gets μ_0 or not. In this lecture we do not go into the problem to construct electromagnetic theory though we adopt $\boldsymbol{E}$-$\boldsymbol{B}$ formulation for the unit of magnetization.

Now we consider a circular current J surrounding the area S in xy-plane as in Fig. 1B.2. The vector potential $\boldsymbol{A}$ is

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\iiint \frac{\boldsymbol{j}}{r}dv = \frac{\mu_0 J}{4\pi}\oint \frac{d\boldsymbol{s}}{r} = \frac{\mu_0 J}{4\pi}\frac{1}{R}\oint\left\{1+\frac{1}{R^2}(\boldsymbol{R}\cdot\boldsymbol{s})+\cdots\right\}$$
$$\simeq \frac{\mu_0 J}{4\pi}\frac{1}{R^3}\oint(\boldsymbol{R}\cdot\boldsymbol{s})d\boldsymbol{s}. \tag{1B.7}$$

s_x component of the integration over $d\boldsymbol{s}$ is

$$\oint(\boldsymbol{R}\cdot\boldsymbol{s})ds_x = \oint\sum_{i=x,y,z}R_i s_i ds_x = \sum_{i=y,z}R_i\oint s_i ds_x,$$



and the term of $R_x s_x$ vanishes because s_x goes back and forth over the integral interval. On the other hand for $s_y ds_x$, $s_x ds_y$, as in the right figure

$$\oint s_y ds_x = -\oint s_x ds_y = -S.$$

This gives

$$\oint(\boldsymbol{R}\cdot\boldsymbol{s})d\boldsymbol{s} = -\oint(\boldsymbol{R}\cdot d\boldsymbol{s})\boldsymbol{s},$$

and from the identities of vector analysis we get

$$\oint(\boldsymbol{R}\cdot\boldsymbol{s})d\boldsymbol{s} = \frac{1}{2}\oint\{(\boldsymbol{R}\cdot\boldsymbol{s})d\boldsymbol{s}-(\boldsymbol{R}\cdot d\boldsymbol{s})\boldsymbol{s}\} = \frac{1}{2}\oint(\boldsymbol{s}\times d\boldsymbol{s})\times\boldsymbol{R}, \tag{1B.8}$$

where

$$\frac{1}{2}\oint\boldsymbol{s}\times d\boldsymbol{s}$$

is the vector perpendicular to the current plane and with the size of the area (S) of the circular current. Then we define a vector $\boldsymbol{\mu}$ as

$$\boldsymbol{\mu} = J\left(\frac{1}{2}\oint\boldsymbol{s}\times d\boldsymbol{s}\right), \tag{1B.9}$$

then we can write

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{\mu}\times\boldsymbol{R}}{R^3}. \tag{1B.10}$$

By using this equation and after some algebra, we obtain, e.g., x-component of $\boldsymbol{B}$ as

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\frac{\mu_0}{4\pi}\left(\nabla\frac{\boldsymbol{\mu}\cdot\boldsymbol{r}}{r^3}\right)_x. \tag{1B.11}$$

Then the magnetic field created by the circular current is given by

$$\boldsymbol{B} = -\frac{\mu_0}{4\pi}\nabla\frac{\boldsymbol{\mu}\cdot\boldsymbol{r}}{r^3}. \tag{1B.12}$$

This is in accordance with eq. (1B.4), and now we know that the circular current is working as a magnetic dipole.

1B.3 Dipole interaction

(Under construction)

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