Lecture note Magnetism (11)

22nd June (2022)   Shingo Katsumoto, Institute for Solid State Physics, University of Tokyo

In the last week we saw an example of magnon BEC in thermal equilibrium observed in a bit special material [*1]. Here we would like to see an experiment on the BEC under quasi-equilibrium condition.

The experimental setup is shown in the left panel of Fig. 5.11[1]. A microwave pulse is applied to an YIG thin film for the generation of low energy ($\sim$ 100 mK) magnons through a parametric process. A photon of microwave has a very small momentum, with which the excitation of a single magnon is difficult. However as shown in the inset, it is possible to excite two magnons with the same momentums but in opposite directions. Therefore, the excited magnons have almost the half of the energy of applied microwave. The concentration and the energy distribution of magnons are measured by Brillouin scattering of a laser light. The right panel of Fig. 5.11 shows the results. With increasing the power of microwave from 4 W to 5.9 W, the number of magnons increased rapidly with keeping the width of distribution. The observed distribution is much broader than the actual one as demonstrated in (d), in which the result of measurement with a higher resolution is displayed. This is claimed as the observation of BEC.

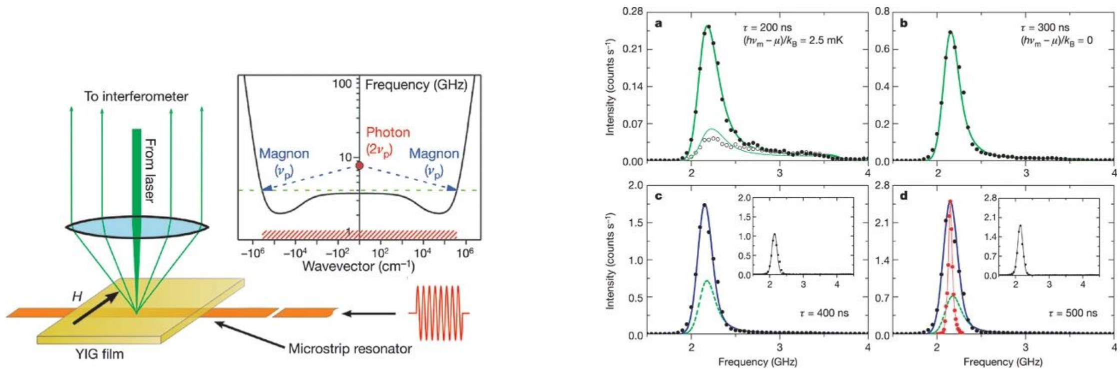

Fig. 5.11 Left: Experimental setup to observe a magnon BEC in quasi-equilibrium. A microwave is applied through a strip line. It excites a number of low energy magnons through a parametric process shown in the inset. A laser beam is applied for measurement of magnon distribution with the Brillouin scattering. Right: Time evolution of energy distribution of magnons after a microwave excitation pulse. Black closed dots are for 5.9 W of the power of microwave. Open ones are for 4 W. The red curve in (d) is with 50 MHz resolution measurement.

5.9.3 Ferromagnetic (Antiferromagnetic) resonance

As noted in Appendix 10A.1, **ferromagnetic resonance** (FMR) is a resonance absorption by the Larmor precession of a macroscopic magnetization. It can also be seen as the long wavelength limit of magnons. In an analysis of an experiment on a system with magnetic order, we need to consider complicated experimental details such as demagnetizing field, which was explained in the first lecture. Let us write the free energy of the system $\mathscr{F}$ as

$$\mathscr{F} = \sum_{\langle i,j \rangle} \lambda_{ij} \boldsymbol{M}_i \cdot \boldsymbol{M}_j - \sum_{i,j} \boldsymbol{M}_i \mathbf{K}_{i,j} \boldsymbol{M}_j - \sum_i \boldsymbol{M}_i \cdot \left(\boldsymbol{H} - \mathrm{N} \sum_j \boldsymbol{M}_j \right), \tag{5.103a}$$

$$= -\lambda \boldsymbol{M}^2 + \boldsymbol{M} \cdot \mathbf{K} \boldsymbol{M} - \boldsymbol{M} \cdot (\boldsymbol{H} - \mathrm{N} \boldsymbol{M}), \tag{5.103b}$$

[*1] Dimer magnetism is not so rare, but it often has a very high critical field of spin gap closing, that situation makes experiments difficult.

where eq. (5.103a) is for a paramagnetic case, and eq. (5.103b) is for a ferromagnetic case with macroscopic magnetization. Though we are mainly using $\boldsymbol{B}$, which is mostly used in experiments, here, we use $\boldsymbol{H}$ to avoid confusion due to existence of a spontaneous magnetization. In the rhs of eq. (5.103a), the first term is the exchange interaction, the second: magnetic anisotropy, the third: Zeeman with demagnetizing effect ($\mathbf{N}$ (tensor), Sec. 1.2.2). Then the effective magnetic field working on $\boldsymbol{M}$ other than $\boldsymbol{H}$ is

$$\boldsymbol{H}_{\text{eff}} = \lambda\boldsymbol{M} - (\boldsymbol{K}+\boldsymbol{N})\boldsymbol{M}, \tag{5.104}$$

in which $\lambda\boldsymbol{M}$ can be dropped from the equation of motion because it is always parallel to $\boldsymbol{M}$. Then the equation of motion is then given by

$$\frac{1}{\gamma}\frac{d\boldsymbol{M}}{dt} = \boldsymbol{M}\times(\boldsymbol{H} - \mathbf{K}\boldsymbol{M} - \mathbf{N}\boldsymbol{M}), \tag{5.105}$$

where γ is the gyromagnetic ratio. This becomes a very complicated form in general experiments. However, in case that the sample shape has an axis of rotational symmetry taken along the easy axis of magnetization, and the external field is also along the easy axis, the resonance frequency ω is given by

$$\omega = \gamma\sqrt{(H + (K_x - K_z + N_x - N_z)M)(H + (K_y - K_z + N_y - N_z)M)}, \tag{5.106}$$

where z-axis is taken along the field.

Also in antiferromagnets, ferrimagnets, similar resonances take place due to the Larmor precession of macroscopic magnetization. In order to treat the situation we consider the resonance conditions of magnetizations $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ [*2] of magnetic sublattices. The effective fields $\boldsymbol{H}_{\text{eff}1}$, $\boldsymbol{H}_{\text{eff}2}$ for $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ respectively, are

$$\boldsymbol{H}_{\text{eff}1} = -\lambda\boldsymbol{M}_2 + \mathbf{K}_{11}\boldsymbol{M}_1 + \mathbf{K}_{12}\boldsymbol{M}_2 + \mathbf{N}(\boldsymbol{M}_1+\boldsymbol{M}_2) \tag{5.107a}$$
$$\boldsymbol{H}_{\text{eff}2} = -\lambda\boldsymbol{M}_1 + \mathbf{K}_{21}\boldsymbol{M}_1 + \mathbf{K}_{22}\boldsymbol{M}_2 + \mathbf{N}(\boldsymbol{M}_1+\boldsymbol{M}_2). \tag{5.107b}$$

In antiferromagnetic case, because $\boldsymbol{M}_1 = -\boldsymbol{M}_2$, the last terms in the two equations of eq. (5.107) vanish. And in the tensor of anisotropy, the followings hold.

$$\mathbf{K}_{11} = \mathbf{K}_{22}, \quad \mathbf{K}_{12} = \mathbf{K}_{21}. \tag{5.108}$$

If the anisotropy is uniaxial, the anisotropic energy $\mathscr{F}_{\text{A}}$ is

$$\mathscr{F}_{\text{A}} = -\frac{K_1}{2}(\cos^2\theta_1 + \cos^2\theta_2), \tag{5.109}$$

where θ_i are the angles between the primary axis and the magnetization $\boldsymbol{M}_i$. The anisotropy tensor is

$$K_{zz} = -\frac{K_1}{|\boldsymbol{M}_1|}, \quad \text{(others)} = 0, \tag{5.110}$$

where the primary axis is taken to z. The resonance conditions, then, are given as follows[2].

$$\frac{\omega_\pm}{\gamma} = \sqrt{2\lambda K_1 + (K_1/|M_1|)^2} \pm H, \qquad H \le H_{\text{c}}, \tag{5.111a}$$
$$\frac{\omega_+}{\gamma} = \sqrt{B^2 - 2\lambda K_1} \qquad H > H_{\text{c}}. \tag{5.111b}$$

Here, $H_{\text{c}} = \sqrt{2\lambda K_1}$ is the critical field of the spin-flop transition. When the anisotropic field $K_1/|\boldsymbol{M}_1|$ is much smaller than the exchange field $\lambda|\boldsymbol{M}_1|$, we can write the condition as

$$\frac{\omega_\pm}{\gamma} = \sqrt{2\lambda K_1} \pm H, \quad H \le H_{\text{c}}. \tag{5.112}$$

In the case of ferrimagnet, the complexities are unavoidable. Refer to [3, 4], if necessary.

[*2] So far we have used $\boldsymbol{M}_{\text{A}}$, $\boldsymbol{M}_{\text{B}}$, which way makes the subscripts awkward. We will use 1, 2 instead, for a while.

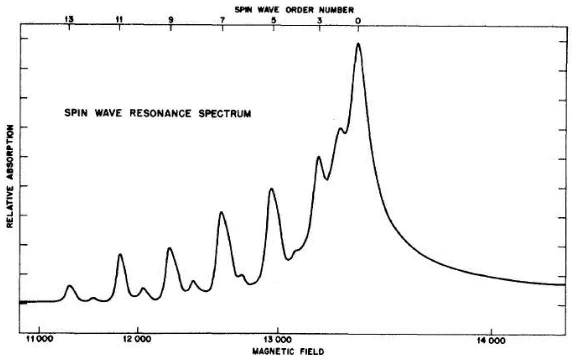

Fig. 5.12 Spin wave resonance (Walker mode) observed in a permalloy film with thickness of 560 nm. The horizontal axis is magnetic field in unit of Oe. The vertical axis is absorption strength for microwave of frequency 8.89 GHz. The boundary condition leaded to the observation of the odd number modes from n=7 to 13 in eq. (5.114). The absorption lines at higher fields were not counted by some reason. From [5].

5.9.4 Spin wave resonance

(A)FMR is a resonance of total magnetization and can be seen as the limit of long wavelength. For thin film samples, standing wave modes perpendicular to the films are expected. The spectra of such standing wave mode is discrete. Those modes are also called **Walker modes** [*3]. From the simplest dispersion relation (5.81) obtained in Secs.5.8.2, 5.8.3, the resonance frequency is

$$\omega_k = \gamma H + \frac{2SJ}{\hbar}(ka)^2. \tag{5.113}$$

Here γ is the gyromagnetic ratio defined for H- representation.

Now, we consider the case that spins are fixed at the surfaces due to some strong magnetic anisotropy. From the boundary condition that the surfaces should be the nodes, the condition of resonance is

$$k = \frac{n\pi}{L}, \quad n = 1, 2, \cdots, \tag{5.114}$$

where L is the film thickness[6]. In a comparatively simple case as the easy axis is perpendicular to the film, various information can be obtained by, e.g. changing the film thickness. This is called **spin wave resonance** (**magnon resonance**). Figure 5.12 shows an example of spectrum. In the era of theoretical proposal and experimental confirmation, the method was frequently used for direct measurement of exchange interaction coefficient J. In recent nano-sized magnets, various confinement shapes have been tested. In many cases, the analysis needs the help of numerical calculations[7].

A resonant motion of macroscopic magnetization can be viewed as a condensate of magnons and this can exists because they are bosons. The phenomenon is, however, different from the BEC, in which the boson system spontaneously condensates breaking the gauge symmetry. On the other hands, resultant phenomena are similar in that many quanta occupy a single quantum state and the condensate behaves as a classical wave. This is, for example, the same for electromagnetic waves, which we are using for communication in daily life. There is no problem in treating these as classical waves, but quantum mechanically, multiple photons are occupying a single quantum state and it is not too much to say they are forming (near) coherent states. This has long been pointed out, e.g., in introductory publications on quantum mechanics[8]. Accordingly, the fact that such condensates of magnons can be created at room temperatures with non-adiabatic methods, is not very surprising. But then, magnon condensate also shows quantum phenomena like quantum tunneling of macroscopic magnetization, or quantum entanglement with photons, and so on. This means we need to be aware of what phenomena we are observing in studying them.

In the lecture, we will see some examples of classical wavy manner in magnon condensates.

[*3] Because the wave equation has the form of Walker equation.

5.10 Renormalization group and scaling theory of phase transitions

Let us be back to general theory of phase transition. Magnetic phase transitions are the best subject for guessing the concept of scaling.

A huge number of review articles and books have been published on the renormalization group (RG), which is a major theme for physics as a whole. Anyone interested in this issue should read, after all, a bible review[9] in this area. As mentioned in the title of this paper, the original motive for developing the renormalization group theory is the Kondo effect discovered in dilute magnetic alloys and the Kondo problem (this is also a major subject for whole physics) raised by Kondo's theory. In that sense, the RG is closely related to the magnetism. I recommend as a basic textbook ref. [10, 11], as a standard for experts ref. [12] [*4]. Among the commentaries on the Internet, ref. [13] seems to be careful and easy to understand. Also a review ref. [14] can be taken in arXiv. In the lecture I would give a super digest explanation.

5.10.1 Correlation function

For the scaling and RG, an important concept is the correlation function. So far we considered the case the magnetization is uniform in space. Even the fluctuations, considered as magnons, have spatially uniform amplitude. However in reality, as we observed in the movie of numerical calculation of Ising model, with lowering the temperature, orderings in spins locally appear (spin **cluster**), the size of which grows on approaching the critical point. If we observe the phenomenon from the view point of spatial fluctuation and in the Fourier space, the characteristic wavelength of fluctuation grows to diverge at the critical point.

Let us treat the above process as follows. We consider a local magnetization density m, weakly depending on $\boldsymbol{r}$ as $m(\boldsymbol{r})$. Let m be a local **order parameter**. By the same logic in Sec. 5.2.1, the free energy density f at $\boldsymbol{r}$ is

$$f(m(\boldsymbol{r},\nabla m(\boldsymbol{r}))=f_0+\frac{a}{2}m^2+\frac{b}{4}m^4+c|\nabla m|^2-hm, \tag{5.115}$$

where in the rhs, the argument $\boldsymbol{r}$ is omitted. h is the local field. Free energy $\mathscr{F}$ is represented in the functional form as

$$\mathscr{F}\{m(\boldsymbol{r})\}=\int_V d\boldsymbol{r}' f(m(\boldsymbol{r}'),\nabla' m(\boldsymbol{r}')). \tag{5.116}$$

The partition function is

$$Z=\int \mathcal{D}m(\boldsymbol{r})\exp\left[-\frac{\mathscr{F}\{m(\boldsymbol{r})\}}{k_{\rm B}T}\right]. \tag{5.117}$$

Here $\mathcal{D}m(\boldsymbol{r})$ is a functional integral, which means taking the sum over all possible $m(\boldsymbol{r})$. This is a well known concept in the Feynman path integral[15, 16]. Because the probability density of realization of distribution $m(\boldsymbol{r})$ is

$$p\{m(\boldsymbol{r})\}=\frac{1}{Z}\exp\left[-\frac{\mathscr{F}\{m(\boldsymbol{r})\}}{k_{\rm B}T}\right], \tag{5.118}$$

the statistical average of a physical quantity A is given by

$$\langle A\rangle=\frac{1}{Z}\int \mathcal{D}m(\boldsymbol{r})A\exp\left[-\frac{\mathscr{F}\{m(\boldsymbol{r})\}}{k_{\rm B}T}\right]. \tag{5.119}$$

Then as we did for eq/ (5.16), we assume the temperature dependences of a and b in eq. (5.115) as

$$a=\alpha(T-T_{\rm C})\quad(\alpha>0),\quad b=\text{const.}\ (>0). \tag{5.120}$$

[*4] I recommend solving the problems. One of my friends in particle field solved all of these. I myself tried some, but felt difficult.

c is also assumed to be constant. We write the **correlation function** of fluctuation in the order parameter as

$$g(\boldsymbol{r}) = \langle (m(0) - \langle m(0) \rangle)(m(\boldsymbol{r}) - \langle m(\boldsymbol{r}) \rangle) \rangle = \langle m(0)m(\boldsymbol{r}) \rangle - \langle m(0) \rangle \langle m(\boldsymbol{r}) \rangle . \tag{5.121}$$

In the region $T > T_{\mathrm{C}}$, the average of the order parameter itself is zero, thus the second term is zero. The Fourier expansion representation of $m(\boldsymbol{r})$ is

$$m(\boldsymbol{r}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} m_{\boldsymbol{k}} \exp(i\boldsymbol{k} \cdot \boldsymbol{r}), \tag{5.122}$$

where V is the system volume. Because $m(\boldsymbol{r})$ is real, $m_{-\boldsymbol{k}} = m_{\boldsymbol{k}}^*$ should hold. In $\langle m(0)m(\boldsymbol{r}) \rangle$, the term $\exp(\pm i\boldsymbol{k} \cdot \boldsymbol{r})$ can be expressed as

$$(m_k + m_{-k})(m_k e^{ikr} + m_{-k} e^{-ikr}) = 2|m_k|^2 e^{-ikr} + 2|m_{-k}|^2 e^{ikr}.$$

By using this and from the translational symmetry of the system, we write

$$g(\boldsymbol{r}) = \frac{1}{V} \sum_{\boldsymbol{k}} \langle |m_{\boldsymbol{k}}|^2 \rangle \exp(-i\boldsymbol{k} \cdot \boldsymbol{r}). \tag{5.123}$$

Then we obtain

$$\mathscr{F} = V f_0 + \sum_{\boldsymbol{k}} |m_{\boldsymbol{k}}|^2 \left(\frac{a}{2} + ck^2 \right) + \frac{b}{4V} \sum_{\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4 = \boldsymbol{0}} m_{\boldsymbol{k}_1} m_{\boldsymbol{k}_2} m_{\boldsymbol{k}_3} m_{\boldsymbol{k}_4}, \tag{5.124}$$

where we take the zero field limit $h \to 0$. In the region $T > T_{\mathrm{C}}$, we can ignore the last 4-th order term in (5.124). The weight function (5.118) now gives the Gaussian distribution

$$\frac{1}{Z} \exp \left[-\frac{2}{k_{\mathrm{B}}T} {\sum_{\boldsymbol{k}}}' \left(\frac{a}{2} + ck^2 \right) (m_{\boldsymbol{k}}^{(\mathrm{r})2} + m_{\boldsymbol{k}}^{(\mathrm{i})2}) \right], \tag{5.125}$$

where we drop the first constant term in eq. (5.124). In the last parentheses, the term $|m_{\boldsymbol{k}}|^2$ is written in a real-imaginary-separated form as

$$\mathrm{Re}[m_{\boldsymbol{k}}] = m_{\boldsymbol{k}}^{(\mathrm{r})}, \quad \mathrm{Im}[m_{\boldsymbol{k}}] = m_{\boldsymbol{k}}^{(\mathrm{i})}.$$

The symbol $\sum_{\boldsymbol{k}}'$ means taking the sum over independent $\boldsymbol{k}$, which give a half of the sum in (5.124). Hence the factor 2 is given. From the above we can write the average of $|m_{\boldsymbol{k}}|^2$ as

$$\langle |m_{\boldsymbol{k}}|^2 \rangle = \frac{k_{\mathrm{B}}T}{a + 2ck^2}. \tag{5.126}$$

Substituting the above into eq. (5.123), $g(\boldsymbol{r})$ is expressed as

$$g(\boldsymbol{r}) = \frac{1}{V} {\sum_{\boldsymbol{k}}}' \frac{k_{\mathrm{B}}T}{a + 2ck^2} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = k_{\mathrm{B}}T \int_0^\infty \frac{e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}}{2ck^2 + a} \frac{d^3k}{(2\pi)^3} = \frac{k_{\mathrm{B}}T}{8\pi d} \frac{\exp(-r/\xi)}{r}, \quad \xi = \sqrt{\frac{2c}{a}}. \tag{5.127}$$

The result indicates that the correlation is damped exponentially on the distance in the region $T > T_{\mathrm{C}}$. ξ in eq. (5.127) is called **correlation length**. Considering the temperature dependence in eq. (5.120), the correlation length varies as $(T - T_{\mathrm{C}})^{1/2}$ in the vicinity of T_{C}. While ξ diverges at T_{C}, the correlation of fluctuation decrease with the distance as $g(r) \propto r^{-1}$.

In the region $T < T_{\mathrm{C}}$, a long range order appears resulting in the difference between the correlation of fluctuation (5.121) and that of the order parameter itself, that is

$$\tilde{g}(\boldsymbol{r}) = \langle m(0)m(\boldsymbol{r}) \rangle . \tag{5.128}$$

This $\tilde{g}(\boldsymbol{r})$ is thus finite for $r \to \infty$. This is equivalent with the appearance of **long range order**[17]. As for $g(\boldsymbol{r})$, it goes zero at $r \to \infty$.

5.10.2 Scaling relations

We introduced the concept of critical exponent in Sec. 5.3. In introducing the scaling relations, we review the "routine" notations. The relevant parameters are the temperature $t \equiv (T - T_C)/T_C$, and the external magnetic field h. The behaviors in the critical region are

$$\begin{aligned}
\text{Specific heat :}\quad & C \sim |t|^{-\alpha}, \\
\text{Magnetization (order parameter) :}\quad & m \sim |t|^{\beta} \quad (t < 0) \\
& m \sim h^{1/\delta} \quad (t = 0), \\
\text{Susceptibility :}\quad & \chi \sim |t|^{-\gamma}, \\
\text{Correlation length :}\quad & \xi \sim |t|^{-\nu}.
\end{aligned}$$

As we saw in the previous subsection, these anomalous behaviors around the critical point $t = h = 0$, originate from the divergence of correlation length, namely the divergence of the size of clusters with a short range order.

The correlation function of fluctuation (5.121) of the system with spatial dimension of d is generally written as

$$g(\boldsymbol{r}) \sim \frac{\exp(-r/\xi)}{r^{d-2+\eta}}. \tag{5.129}$$

Here η is one of the critical exponents and in eq. (5.127) (i.e., the GL theory) $\eta = 0$. The GL theory also gives

$$\alpha = 0, \quad \beta = 1/2, \quad \gamma = 1, \quad \delta = 3, \quad \nu = 1/2, \quad \eta = 0. \tag{5.130}$$

The GL theory is a phenomenology. The critical exponents depend on the "universality" of the system as we already saw.

These critical exponents have the following relations.

$$\gamma = (2 - \eta)\nu, \tag{5.131a}$$
$$\alpha + 2\beta + \gamma = 2, \tag{5.131b}$$
$$\beta + \gamma = \beta\delta. \tag{5.131c}$$

Among them, eq. (5.131b) and eq. (5.131c) are called **scaling relation**, which can be derived from the **scaling ansatz** explained in the following. We have taken t and h as the relevant parameters of the transition, but the ansatz tells the behavior of the system in the vicinity of transition is determined by single relevant parameter $h/|t|^{\Delta}$. Here Δ is called **gap exponent**.

We assume that the anomalous part of the free energy f_s can be expressed with the use of a function $f_{\pm}(x)$

$$f_s \sim |t|^{2-\alpha} f_{\pm}\left(\frac{h}{|t|^{\Delta}}\right), \tag{5.132}$$

around the critical point. $\pm$ in $f_{\pm}$ correspond to $T \gtrless T_C$. For $h = 0$, t can move without affecting $h/|t|^{\Delta}$. The factor $|t|^{2-\alpha}$ is then attached to have the consistency with the specific heat. Now the magnetization and the susceptibility are

$$m(h = 0) \sim -\frac{\partial f_s}{\partial h} \sim |t|^{2-\alpha-\Delta} f'_{\pm}(0) \sim |t|^{\beta} \quad (t < 0) \tag{5.133}$$
$$\chi \sim -\frac{\partial^2 f}{\partial h^2} \sim |t|^{2-\alpha-2\Delta} f''_{\pm}(0) \sim |t|^{-\gamma}. \tag{5.134}$$

These lead to the relations

$$\beta = 2 - \alpha - \Delta \tag{5.135}$$
$$-\gamma = 2 - \alpha - 2\Delta. \tag{5.136}$$

Then erasing

$$\Delta = \beta + \gamma \tag{5.137}$$

from the above, a scaling relation (5.131b) is obtained.

Next to see the behavior of magnetization at $t = 0$, we need to explore the asymptotic behavior of $f'_\pm$ for $h/t^\Delta \to \infty$. We assume the asymptotic form as

$$f'_\pm(x) \sim x^{\lambda_\pm} \quad (x \to \infty). \tag{5.138}$$

Then the asymptotic behavior of eq. (5.133) is

$$m \sim |t|^\beta f'_\pm\left(\frac{h}{|t|^\Delta}\right) \sim \frac{h^{\lambda_\pm}}{|t|^{\Delta\lambda_\pm - \beta}}. \tag{5.139}$$

For m to be finite for $t \to 0$,

$$\lambda_\pm = \frac{\beta}{\Delta} = \frac{\beta}{\beta+\gamma}. \tag{5.140}$$

From this relation and the relation and the definition of the critical exponent $m \sim h^{1/\delta}$,

$$\delta = \frac{\beta+\gamma}{\beta}, \tag{5.141}$$

which is eq. (5.131c). There is also a relation

$$2 - \alpha = d\nu, \tag{5.142}$$

with the spatial dimension d, and is called hyperscaling relation.

5.10.3 Renormalization group

As approaching the critical point, the correlation length of fluctuation ξ gets longer. Within the space of this length, some order is growing. Taking a distance x shorter than ξ ($1 < x \ll \xi$), we can assume that the system is uniform in a space with a size of x. We thus average physical quantities within the size x and then the unit of length is changed to x (i.e., $x \mapsto 1$). This is called coarse graining. With this operation, the correlation length is transformed to ξ/x and the system looks being drew apart from the critical point. Similarly, the parameters of the system get various modifications by this operation. This operation is called **renormalization group (RG) transformation** with scaling factor x.

Let $\mathcal{R}(x)$ be such an operation, then, e.g, the transform of $\mathscr{H} \to \mathscr{H}'$ can be written as

$$\mathscr{H}' = \mathcal{R}(x)\mathscr{H}. \tag{5.143}$$

Sequential operation of $\mathcal{R}(x)$ and $\mathcal{R}(x')$ is the same as a single RG transform with scaling factor of xx'. That is

$$\mathcal{R}(x')\mathcal{R}(x) = \mathcal{R}(x'x). \tag{5.144}$$

The RG transformation thus fulfils the associativity forming a semigroup. It is called then **renormalization group**. Generally there is no inverse transformation from once coarse-grained system to the original fine-grained system. Hence a set of such transformations is not a group but semigroup.

Figure 5.13(a) show an example of an Ising model on a two-dimensional square lattice, in which four spins are averaged into one as

$$s_q = \frac{1}{4}\sum_i s_{qi}. \tag{5.145}$$

Then a new lattice in Fig. 5.13(b) appears. The scaling factor is $\sqrt{4} = 2$. The averaging makes the spins to be able to take the value of ± 1, and the model is no longer an Ising model. Also the range of interaction is modified. Accordingly, with repetition of RG transformation, the system moves in the parameter(s) space. The invariants with RG transformation are the symmetry of the order parameter and the spatial dimension. The parameter(s) space is then characterized by them.

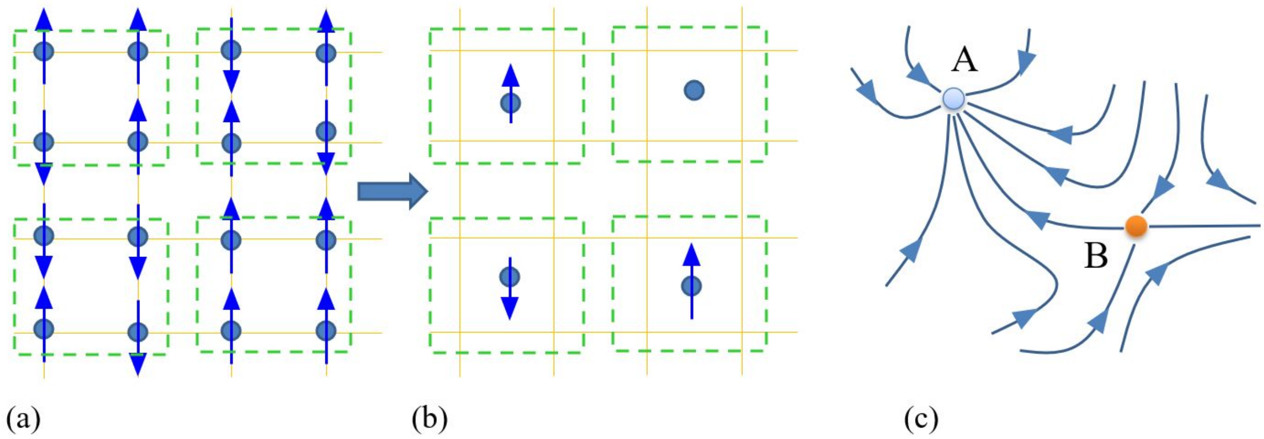

Fig. 5.13 (a) Illustration of an Ising model on two-dimensional square lattice. (b) Schematically shows a renormalization group transformation of averaging four spins in (a) as a single spin. (c) Schematic illustration of a flow diagram. A is a stable fixed point. B is an unstable fixed point.

Now we take the scaling factor as a continuous variable. Then the system transitions in the parameter space become continuous. Figure 5.13(c) shows an example of such continuous "flows" of system in parameter space by RG transformation. This kind of diagram is called **flow diagram**. If a system is in the high temperature side of the critical temperature at the starting point, the RG transformation drives it to the completely disordered state. If the starting point is in the opposite lower side of $T_{\rm C}$ ($t=0$), it will transition to the perfectly ordered state. Perfectly ordered/disordered states are invariant for RG transformation, hence they form fixed points in the flow diagram as illustrated in Fig. 5.13(c). They are called stable fixed points, an example of which is indicated as A in Fig. 5.13(c). They collect the flow lines. On the other hand, just on the critical point, the correlation length of fluctuation diverges and the system on it does not change with RG transformation. Therefore the critical point is a fixed point. However, an infinitesimally small shift in the parameter causes repelling of flow lines from the critical point. Hence it is called unstable fixed point. In Fig. 5.13(c), B is an example of such an unstable fixed point.

5.10.4 Derivation of scaling ansatz

We again take t (temperature) and h (magnetic field) as the relevant parameters of transition. We write a RG transformation with a scale factor x as

$$t' = g_1^{(x)}(t,h), \tag{5.146a}$$

$$h' = g_2^{(x)}(t,h). \tag{5.146b}$$

In the vicinity of the fixed point $t=h=0$ (critical point), we try to expand the above in the form

$$\begin{aligned} t' &\simeq \Lambda_{11}(x)t + \Lambda_{21}(x)h, \\ h' &\simeq \Lambda_{21}(x)t + \Lambda_{22}(x)h. \end{aligned} \tag{5.147}$$

There is no constant (0-th order) term because it is a fixed point. We further know that there is no linear coupling between t and h because the sign of h is reversed with reversing the sign of the order parameter while t does not change with that. Namely $\Lambda_{12}(x)=\Lambda_{21}(x)=0$. Therefore the association law (5.144) gives

$$(\Lambda_{11}(x))^n = \Lambda_{11}(x^n), \quad (\Lambda_{22}(x))^n = \Lambda_{22}(x^n). \tag{5.148}$$

This should hold for any natural number n and any x (>1). This means $\Lambda_{11}(x)$ and $\Lambda_{22}(x)$ are power functions of x.

$$\Lambda_{11}(x) = x^{\lambda_1}, \quad \Lambda_{22}(x) = x^{\lambda_2}. \tag{5.149}$$

Now we apply a RG transform with scaling factor x, n-times on a system that has the starting point at (t,h). We assume the final state temperature $t_0 = x^{n\lambda_1} t$ is far enough from the critical point. Because the correlation length of the fluctuation is

$$\frac{\xi(t)}{\xi(t_0)} = x^n = \left(\frac{t}{t_0}\right)^{-1/\lambda_1}, \tag{5.150}$$

and from the critical exponent definition $\xi \sim |t|^{-\nu}$, we obtaine $\nu = \lambda_1^{-1}$.

On the other hand in d-dimensional system, the free energy density $f(t,h)$ becomes x^d times by the RG transformation. Therefore

$$x^{nd} f(t,h) = f(x^{n\lambda_1} t, x^{n\lambda_2} h) = f(t_0, (t/t_0)^{-\lambda_2/\lambda_1} h). \tag{5.151}$$

We see t_0 as a constant, then with an appropriate function $f_\pm(x)$ we write

$$f(t,h) = t^{d/\lambda_1} f_\pm(t^{-\lambda_2/\lambda_1} h) = t^{d\nu} f_\pm\left(\frac{h}{t^\Delta}\right) \quad \Delta = \frac{\lambda_2}{\lambda_1}. \tag{5.152}$$

This is showing the scaling ansatz.

Chapter 6
Magnetism of itinerant electrons

The mechanism of ferromagnetism in common metals such as iron, cobalt, and nickel is thought to be significantly different from what we have seen so far.

References

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