Physics of Semiconductors

12th 2016.7.4

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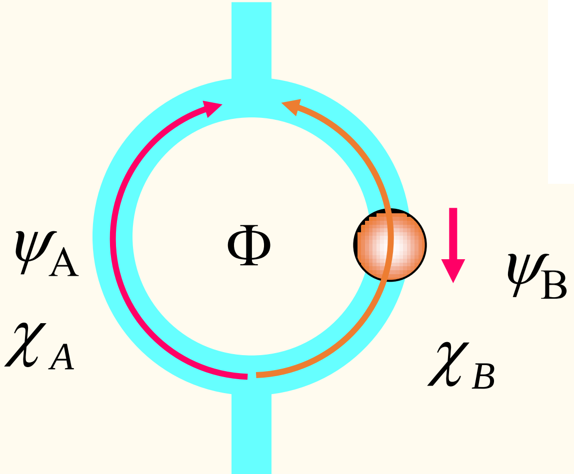
Outline today

- The Kondo effect in quantum dots

Quantum Hall effects

- Integer quantum Hall effect
- Edge mode explanation
- Topological aspect
- Edge-Bulk correspondence
- Fractional quantum Hall effect

The Kondo effect

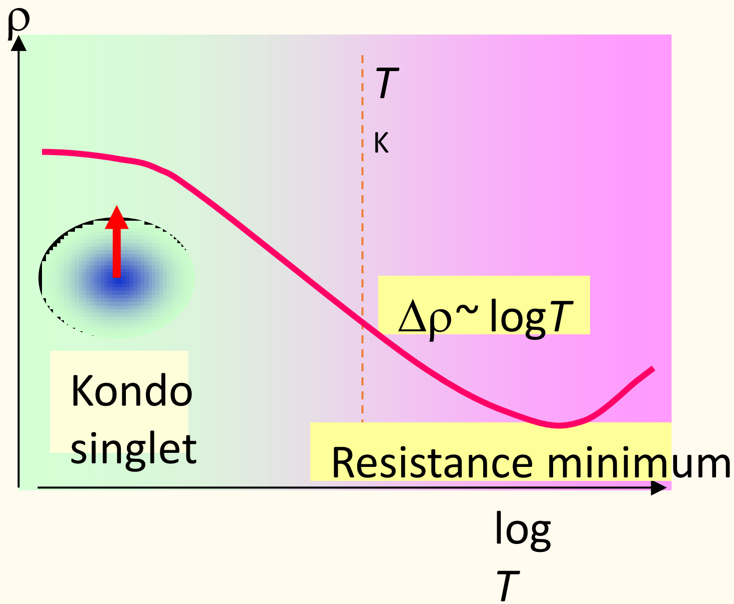


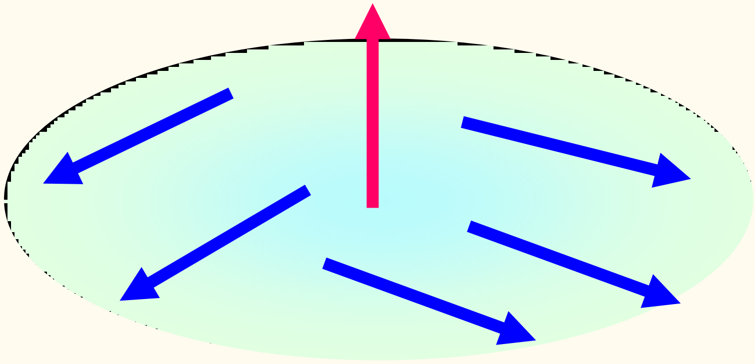
$$\psi_A : |\psi_A \uparrow\rangle|d\downarrow\rangle$$

$$\psi_B : \frac{1}{\sqrt{2}}\left(|\psi_B \uparrow\rangle|d\downarrow\rangle - |\psi_B \downarrow\rangle|d\uparrow\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|s\uparrow\rangle|d\downarrow\rangle - |s\downarrow\rangle|d\uparrow\rangle\right)$$

Jun Kondo





Kondo singlet

A. Yoshimori, PR168 (1967)

Kondo state:

$$\psi = \{ \sum_k [\Gamma_k{}^\alpha a_{k\downarrow}{}^\dagger \alpha + \Gamma_k{}^\beta a_{k\uparrow}{}^\dagger \beta]$$

$$+ \sum_{k_1k_2k_3} [\Gamma_{k_1k_2k_3}{}^{\alpha\downarrow} a_{k_1\downarrow}{}^\dagger a_{k_2\downarrow}{}^\dagger a_{k_3\downarrow}\alpha + \Gamma_{k_1k_2k_3}{}^{\beta\uparrow} a_{k_1\uparrow}{}^\dagger a_{k_2\uparrow}{}^\dagger a_{k_3\uparrow}\beta$$

$$+ \Gamma_{k_1k_2k_3}{}^{\alpha\uparrow} a_{k_1\downarrow}{}^\dagger a_{k_2\uparrow}{}^\dagger a_{k_3\uparrow}\alpha + \Gamma_{k_1k_2k_3}{}^{\beta\downarrow} a_{k_1\uparrow}{}^\dagger a_{k_2\downarrow}{}^\dagger a_{k_3\downarrow}\beta]$$

$$+ \cdots \} \psi_v, \quad (1)$$

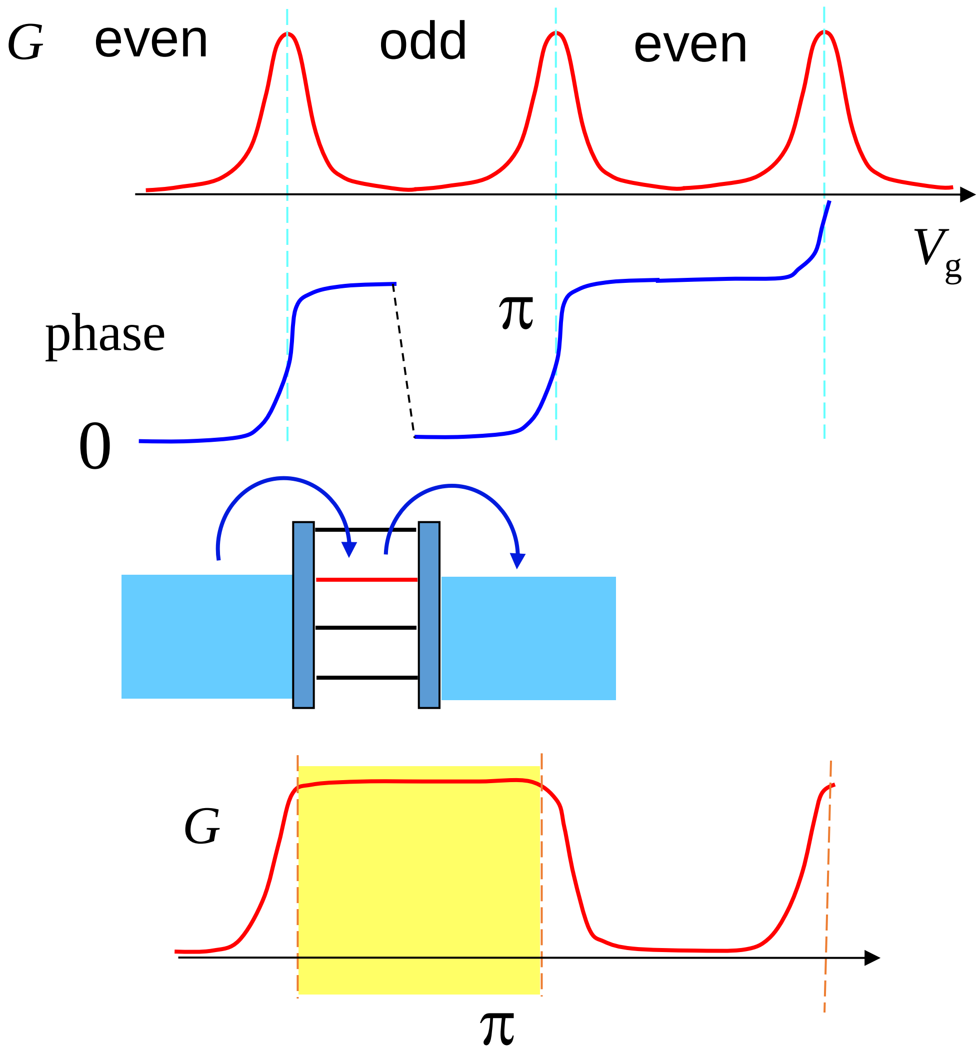
Yosida's variational ground state

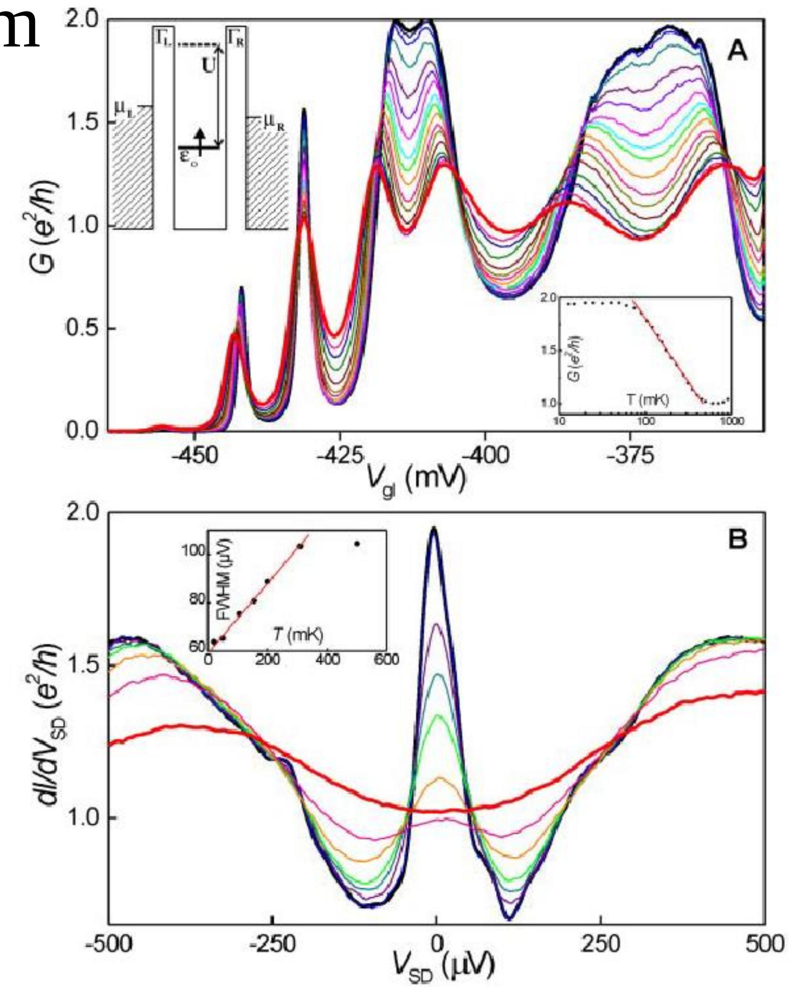
Fermi sea state

Many body resonance → multiple scattering with many electrons of the same energy (Fermi energy) with quantum entanglement in spin

→ Spatially localized state, energy level is the same as the Fermi energy

Kondo effect in a quantum dot system

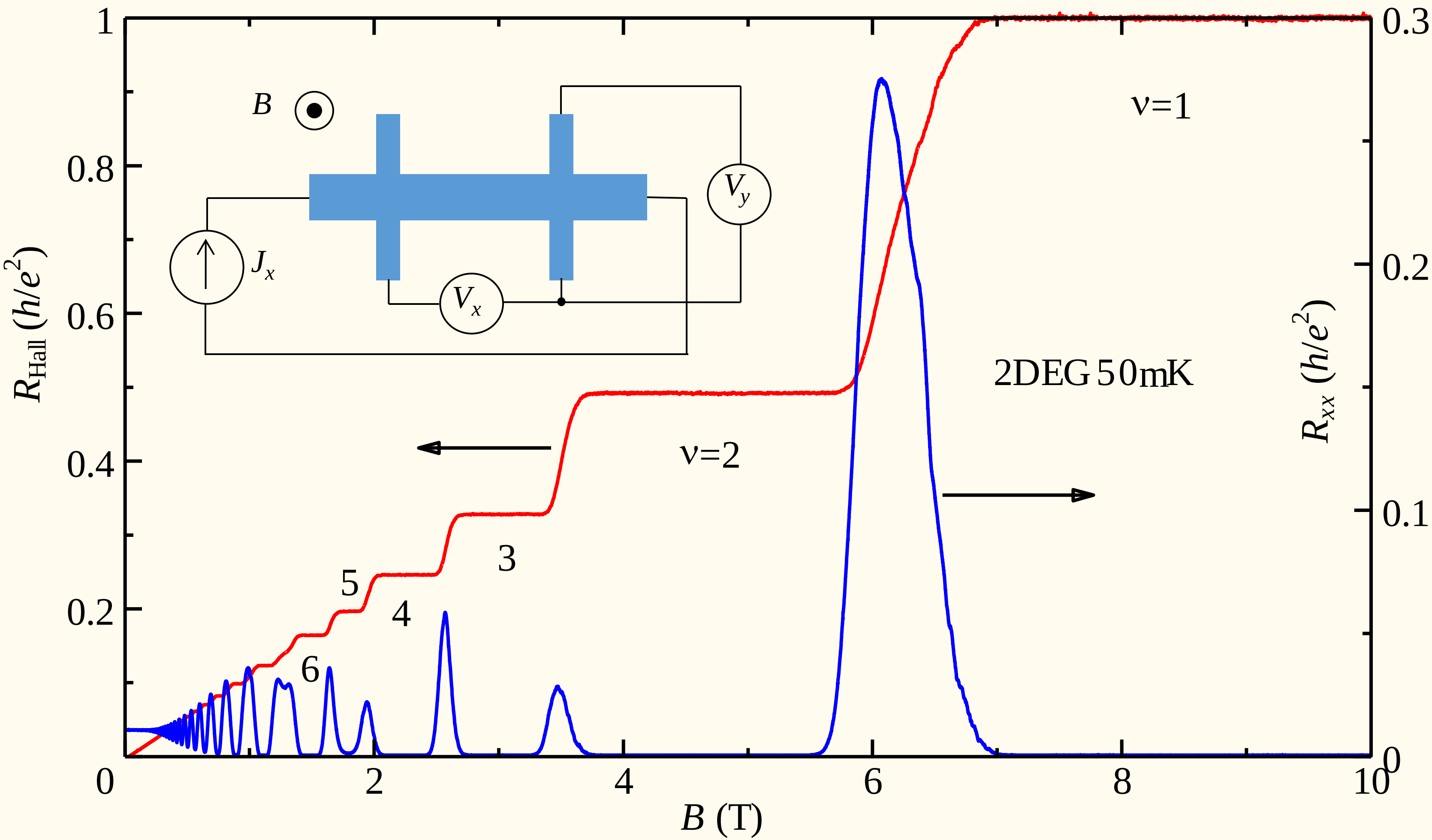




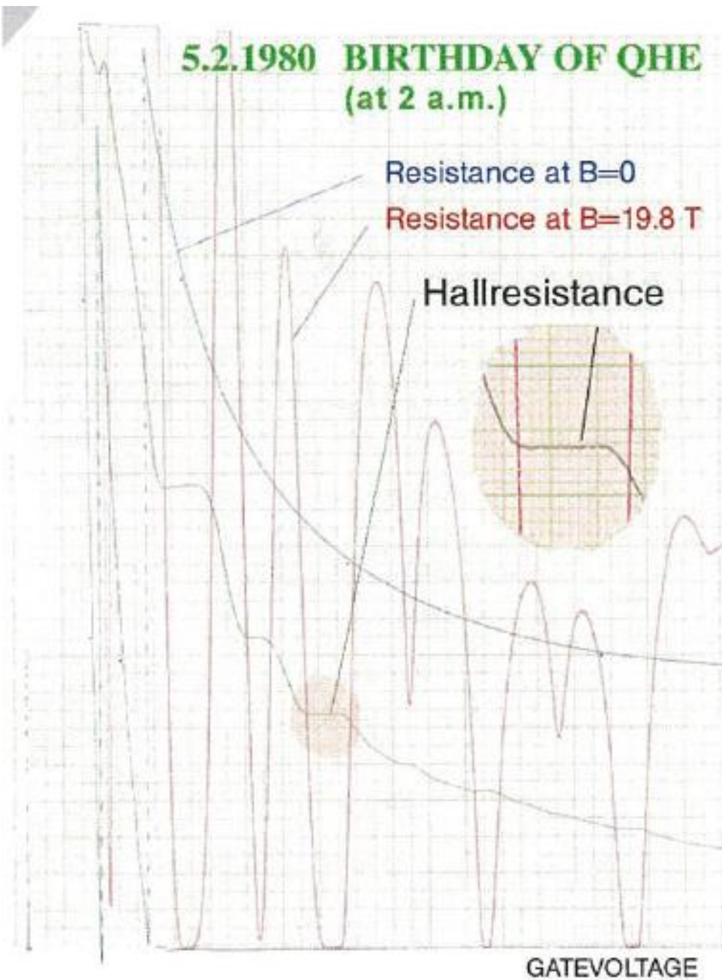
W. G. van der Wiel et al.
Science **289**, 2105 (2000).

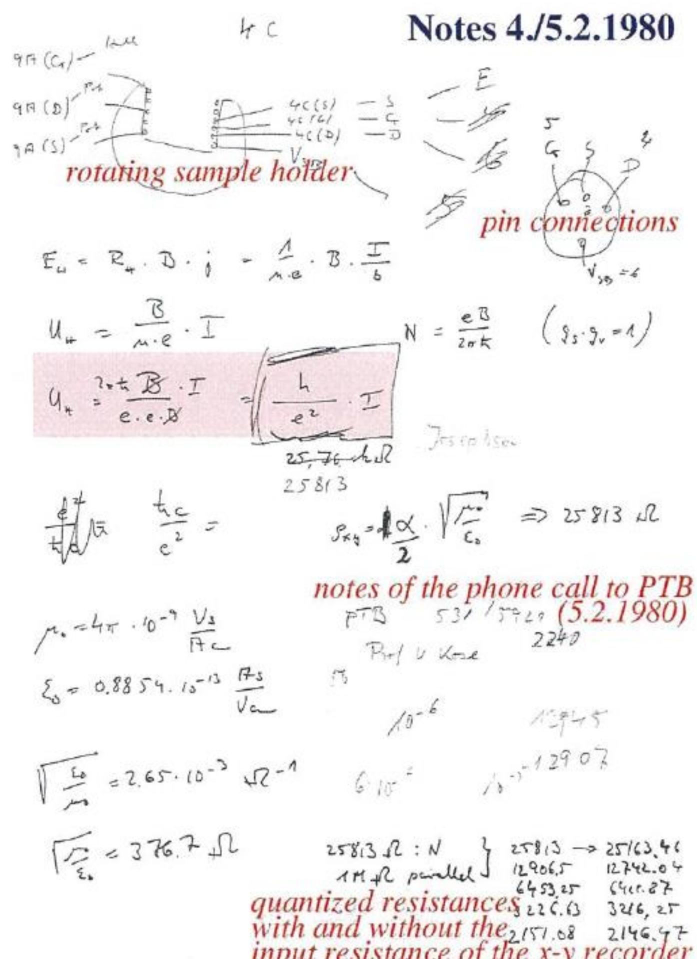

Ch.6 Quantum Hall Effects
WIDEWALLPAPERS.NET

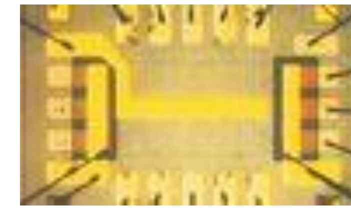
Integer Quantum Hall Effect



Birthday of quantum Hall effect







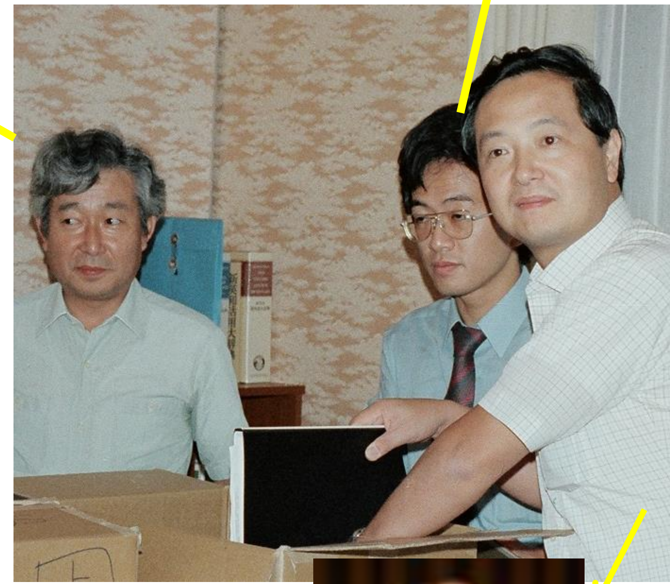
Klaus von Klitzing

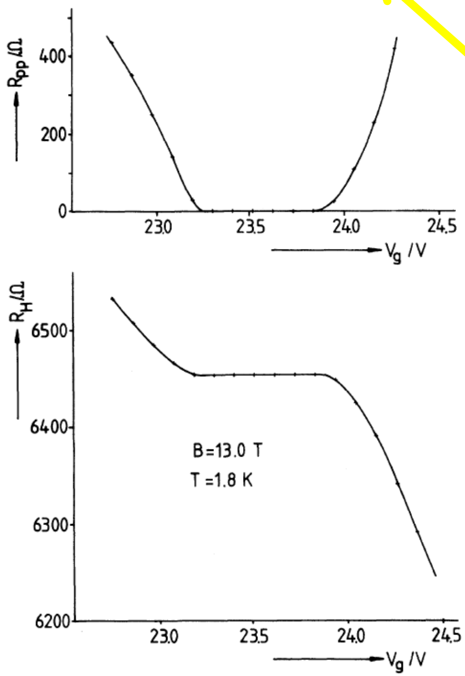
Discovery of quantum Hall effect

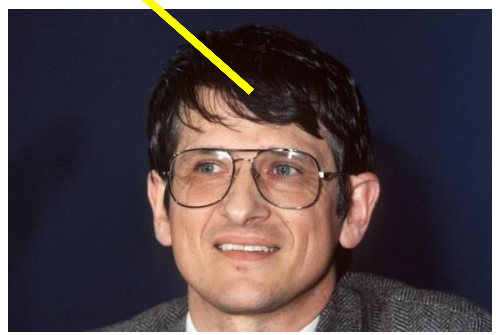
Klaus von Klitzing

Shinji Kawaji

Yasuhiro Iye



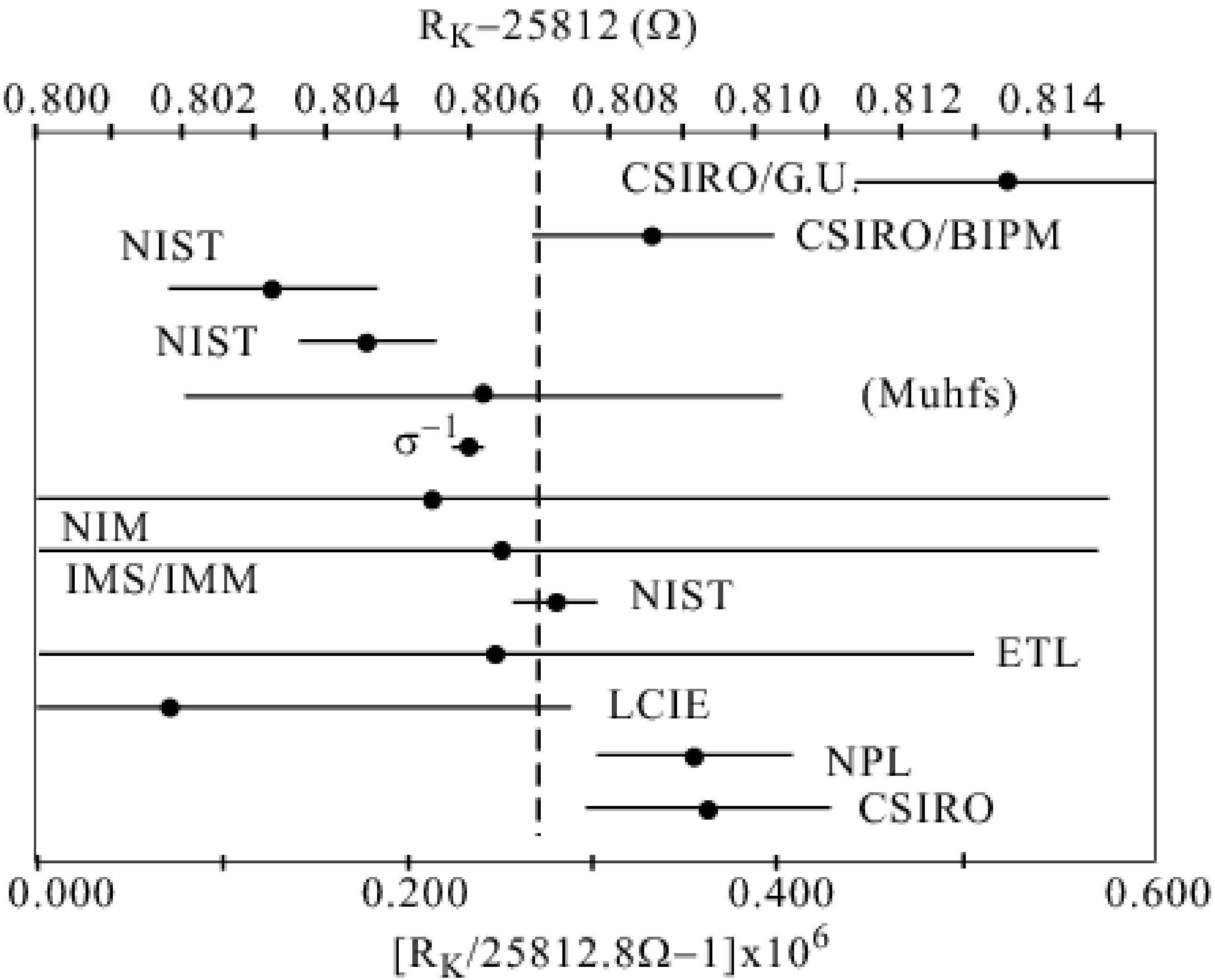




Phys. Rev. Lett. **45**, 494 (1980)

Tsuneya Ando

Standard of resistance



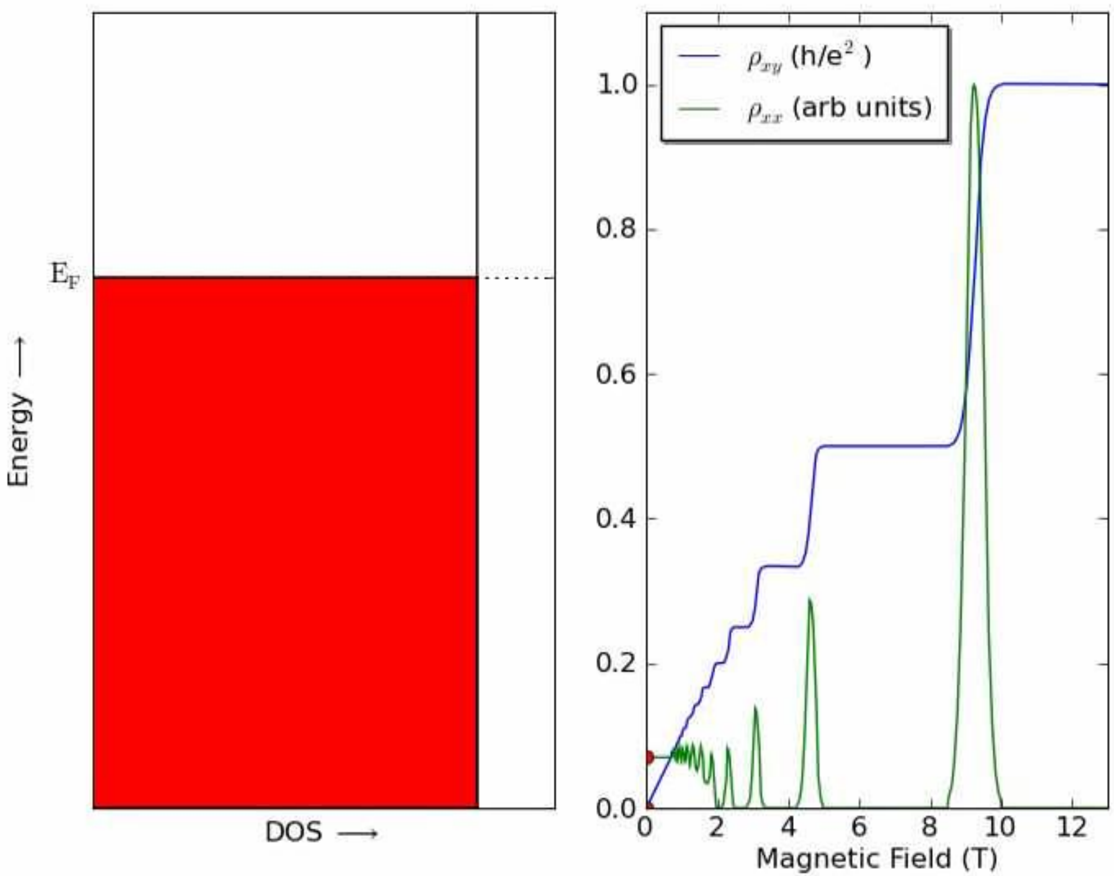
$$R_{\mathrm{K}} = \frac{h}{e^2}$$

von Klitzing constant

25812.8074434(84) V/A

$\pm 3.2 \times 10^{-10}$

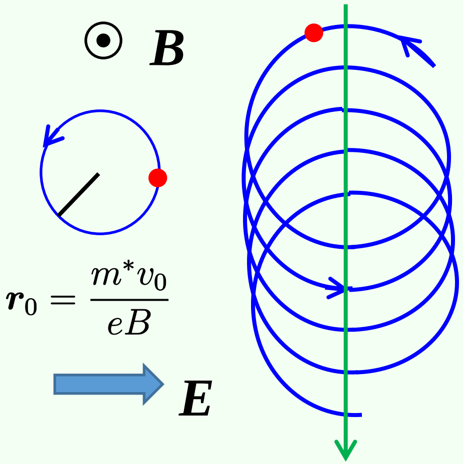
IQHE and Landau quantization



From Wikipedia

Two dimensional electrons under magnetic field

Lorentz force $m^*\ddot{\boldsymbol{r}} = -e\boldsymbol{v}\times\boldsymbol{B}$



$$\boldsymbol{r} = \boldsymbol{R} + r_0(\cos\omega_{\rm c}t, \sin\omega_{\rm c}t)$$

$$\omega_{\rm c} \equiv \frac{eB}{m^*},\ r_0 \equiv \frac{v_0}{\omega_{\rm c}}$$ Cyclotron frequency, radius

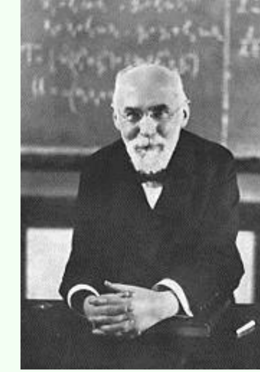
$\boldsymbol{R}$: Guiding center

$$m^*\ddot{\boldsymbol{r}} = -e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

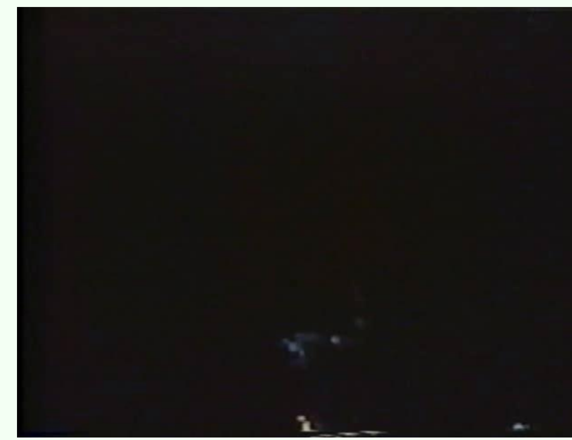
$\boldsymbol{R}$: Moves vertically to $\boldsymbol{E}$ with constant velocity E/B

$$\mathscr{H} = \frac{m}{2}\boldsymbol{v}^2 = \frac{(\boldsymbol{p}+e\boldsymbol{A})^2}{2m} \qquad \boldsymbol{B} = {\rm rot}\,\boldsymbol{A}$$

$$\equiv \frac{\boldsymbol{\pi}^2}{2m} = \frac{\pi_x^2 + \pi_y^2}{2m} \qquad \boldsymbol{\pi} \equiv \boldsymbol{p} + e\boldsymbol{A}$$



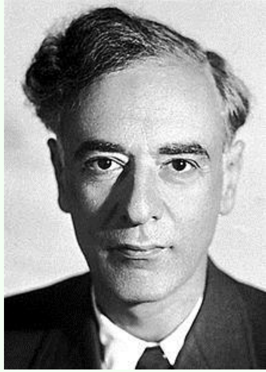
Hendrik Lorentz
1853 - 1928



Landau quantization

$$[\pi_\alpha, \beta] = -i\hbar\delta_{\alpha\beta}\ (\alpha, \beta = x, y), \quad [\pi_x, \pi_y] = -i\frac{\hbar^2}{l^2}$$

$$l \equiv \sqrt{\frac{\hbar}{eB}} = \sqrt{\frac{1}{2}}\sqrt{\frac{\phi_0}{\pi B}}$$ Magnetic length



Lev Landau
1908 - 1968

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{R}} + \frac{l^2}{\hbar}(\pi_y, -\pi_x) \qquad \boldsymbol{R} = (X, Y) \quad [X, Y] = il^2$$

$$a = \frac{l}{\sqrt{2}\hbar}(\pi_x - i\pi_y), \quad a^\dagger = \frac{l}{\sqrt{2}\hbar}(\pi_x + i\pi_y)$$

$$[a, a^\dagger] = 1, \quad \mathscr{H} = \hbar\omega_c\left(a^\dagger a + \frac{1}{2}\right) \quad E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)\ (n = 0, 1, 2, \cdots)$$

Landau quantization: Landau gauge

Diagonalize X : Landau gauge $\boldsymbol{A} = (0, Bx)$

$$\mathscr{H}\psi = \frac{(\boldsymbol{p}+e\boldsymbol{A})^2}{2m}\psi = \frac{-1}{2m}\left[\frac{\hbar^2\partial^2}{\partial x^2} - \left(-i\frac{\hbar\partial}{\partial y} + eBx\right)^2\right]\psi(\boldsymbol{r})$$

$$= \frac{1}{2m}\left[-\hbar^2\nabla^2 - 2i\hbar eBx\frac{\partial}{\partial y} + e^2B^2x^2\right]\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r})$$

$$\psi(\boldsymbol{r}) = u(x)\exp(iky)$$

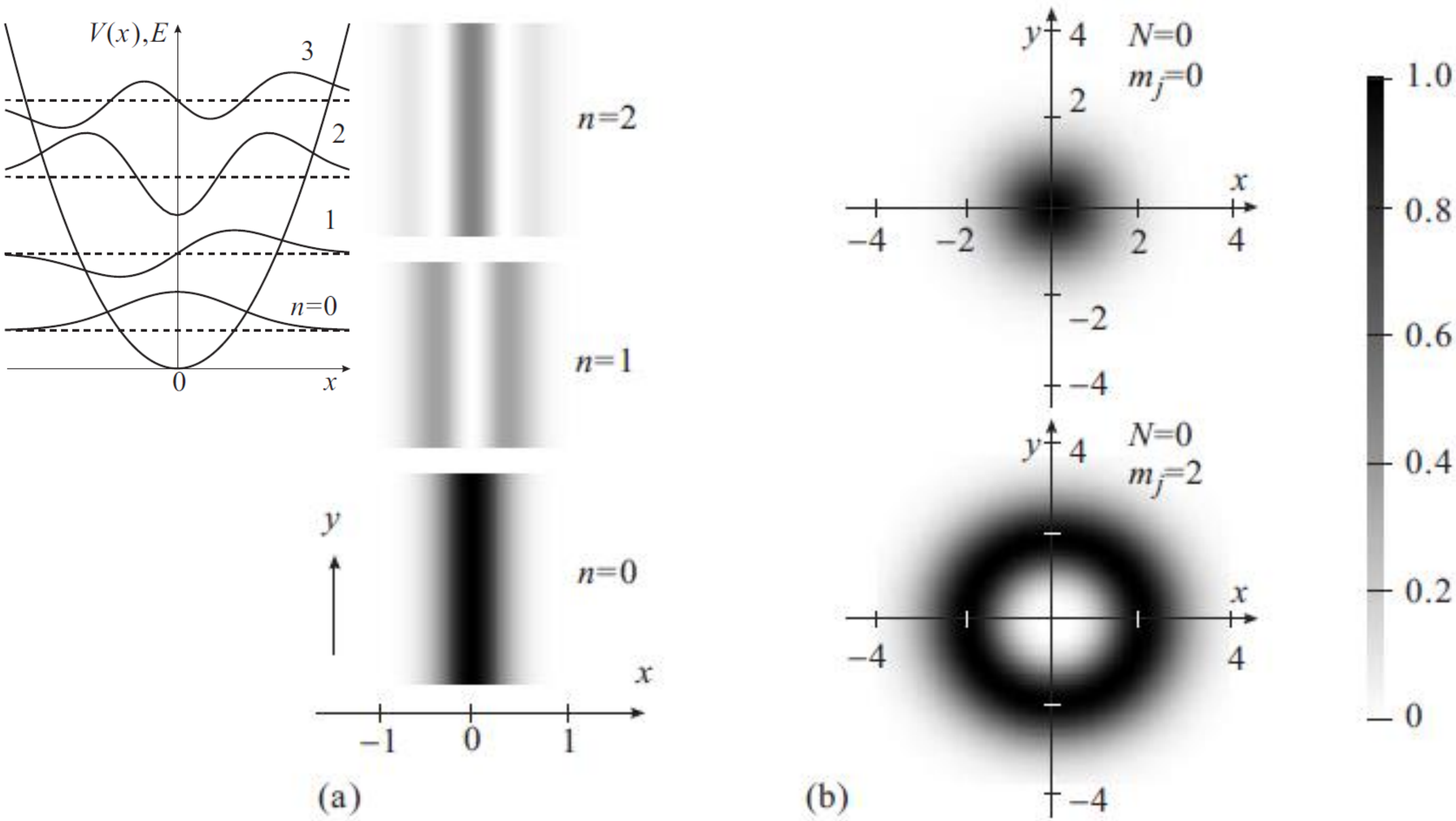
$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{(eB)^2}{2m}\left(x+\frac{\hbar}{eB}k\right)^2\right]u(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega_c^2}{2}(x+l^2k)^2\right]u(x) = Eu(x)$$

$$\psi_{nk}(\boldsymbol{r}) \propto H_n\left(\frac{x-x_k}{l}\right)\exp\left(-\frac{(x-x_k)^2}{2l^2}\right)\exp(iky) \quad (x_k \equiv -l^2k)$$

$$X = x_k = -l^2k = -l^2p_y/\hbar \qquad \frac{dE}{dk} = 0$$

Group velocity = 0

Landau quantization: forms of wavefunctions



Diagonalize X

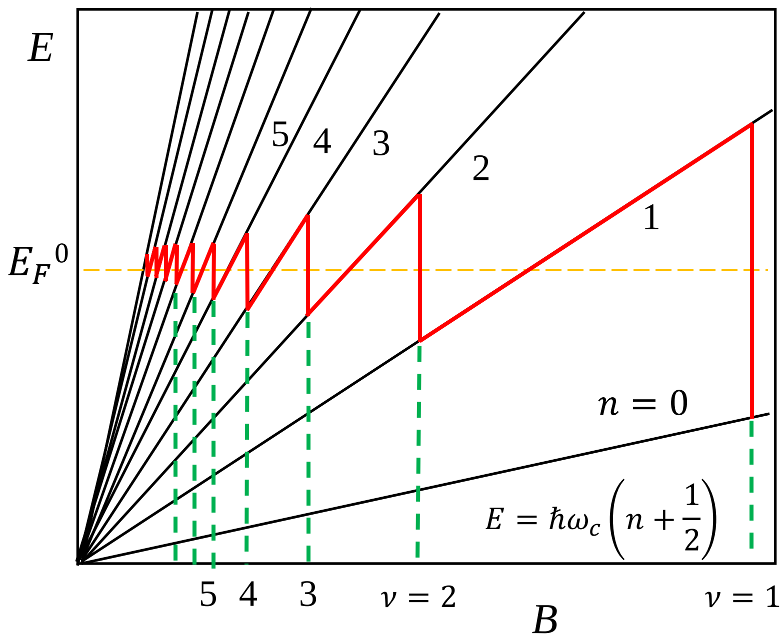
Diagonalize $X^2 + Y^2$ ← Symmetric gauge $\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{r}/2$

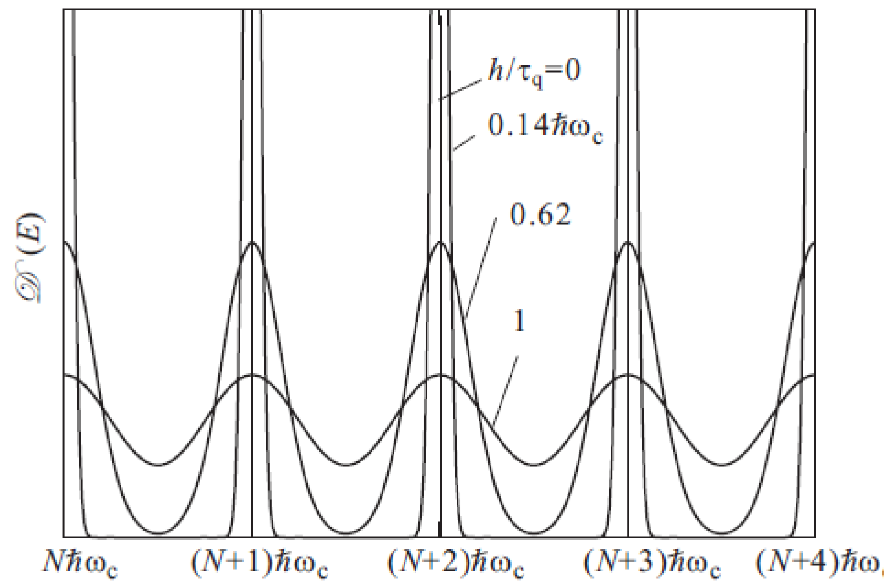
Shubnikov-de Haas oscillation

Number of states in $S = W_x \times W_y$ $\quad 0 \leq X \leq W_x \rightarrow -W_x l^2 \leq k \leq 0$

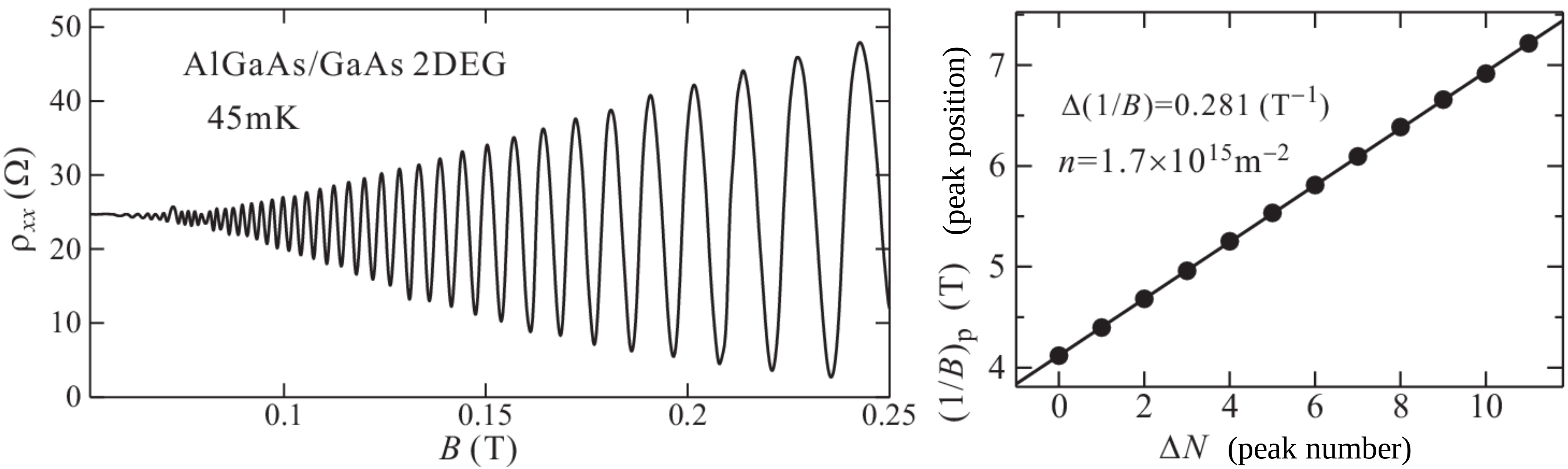
"Distance" of k-values in y-direction: $2\pi/W_y$ $\quad \dfrac{W_x/l_B^2}{2\pi/W_y} = \dfrac{S}{2\pi l_B^2}$ $\quad n_{\rm L} = \dfrac{1}{2\pi l_B^2} = \dfrac{eB}{h} = \dfrac{B}{\phi_0}$

$$\nu = \frac{\phi_0 n_s}{B}$$: Filling factor (number of Landau levels filled with electrons)



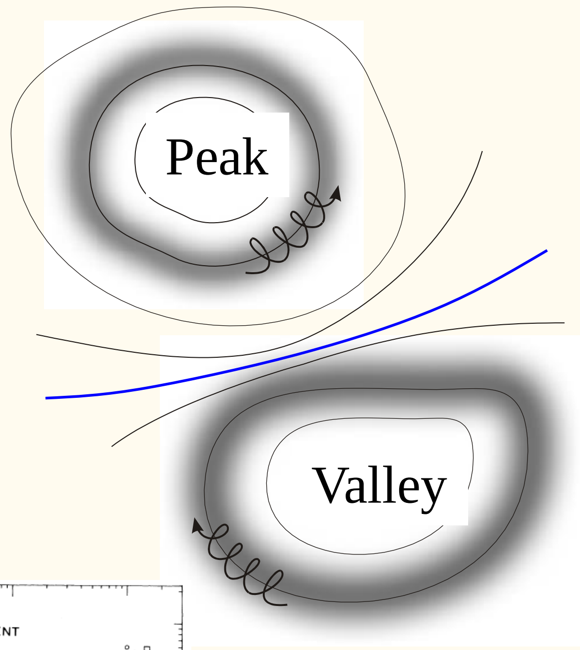


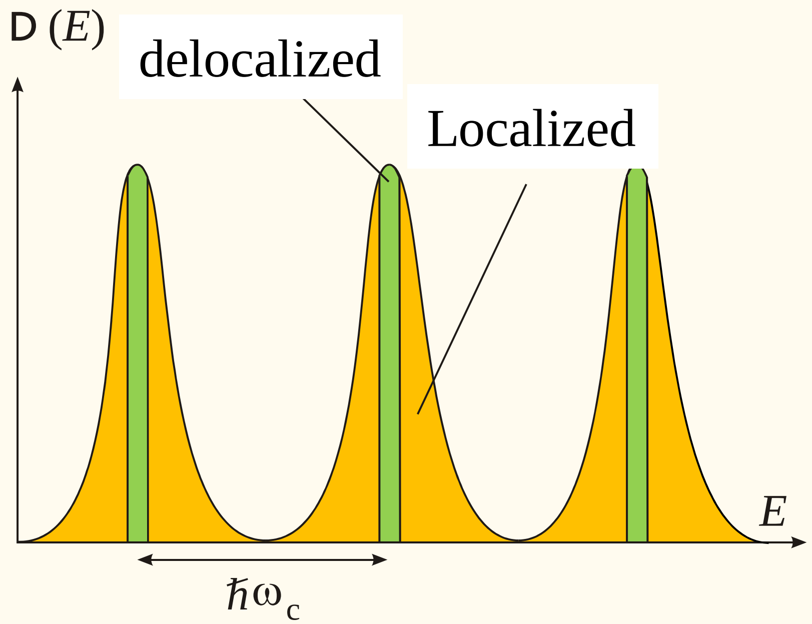
SdH oscillation (example)

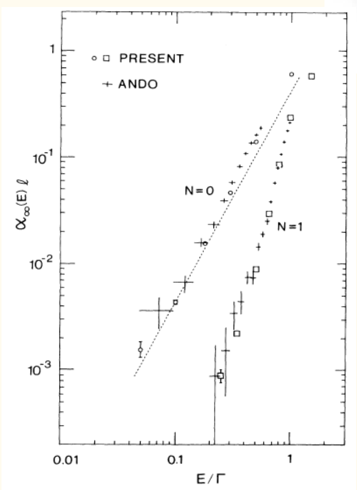


$$n = \frac{2}{\phi_0 \Delta(1/B)} = \frac{4.83 \times 10^{14}}{\Delta(1/B)} \ \ (\mathrm{m}^{-2})$$

Localization/delocalization of wavefunctions







$$\xi(E)^{-1} = \alpha(E) \propto |E - E_N|^s$$

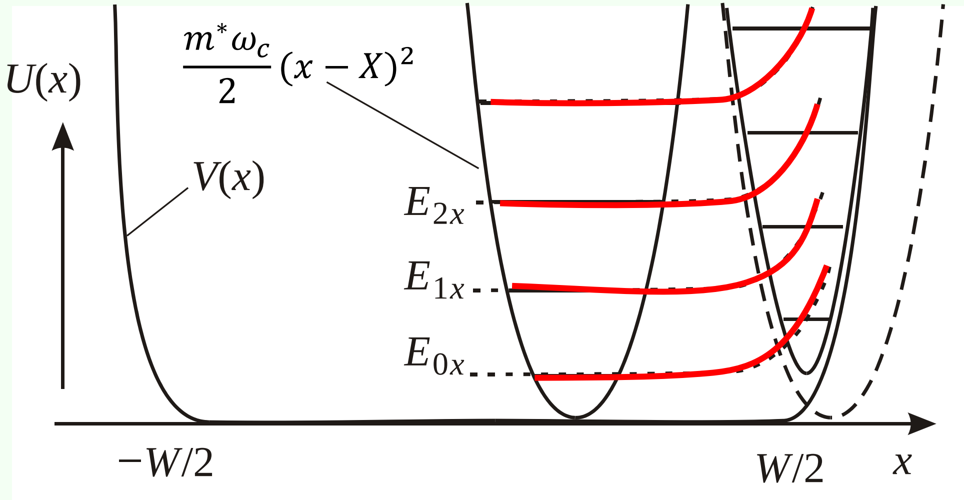
Numerical simulation $s = 2$ for $N = 0$

Aoki & Ando, PRL **54**, 831 (1985).

Edge mode explanation of IQHE

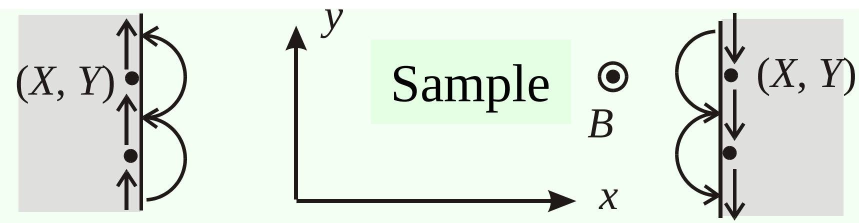
Markus Büttiker
1950-2013

$$\langle v_y \rangle = \frac{dE}{\hbar dk} = -\frac{l_B^2}{\hbar}\frac{dE}{dX}$$



Current brought by a Landau edge mode

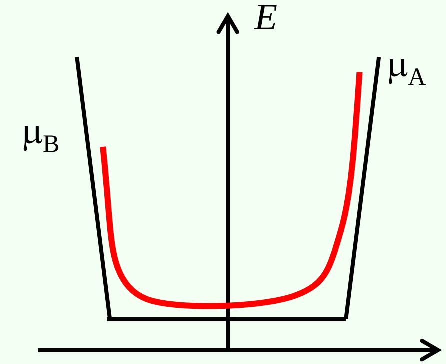
$$J = \int_{X_0}^{X_\mu} \frac{L_y dX}{2\pi l_B^2}\frac{e}{L_y}\langle v_y \rangle$$
$$= \frac{e}{h}\int \mathrm{d}X \frac{\mathrm{d}E}{\mathrm{d}X} = \frac{e}{h}(\mu - E_0)$$



One dimensional system:

Landauer formula is applicable

$$\sigma_{xy} = \frac{J_y}{V_x} = \frac{e(J_\mathrm{A} - J_\mathrm{B})}{\mu_\mathrm{A} - \mu_\mathrm{B}} = \frac{e^2}{h}$$



Chiral edge mode: No backscattering!

Explanation from topological aspect


David Thouless


Mahito Kohmoto

Bloch electrons under magnetic field: tight binding model

Translational operator: $T_{\boldsymbol{R}} f(\boldsymbol{r}) = f(\boldsymbol{r}+\boldsymbol{R}), \quad T_{\boldsymbol{R}} = \exp\left(\frac{i}{\hbar}\boldsymbol{R}\cdot\boldsymbol{p}\right)$

Hamiltonian: $\mathscr{H}_0 = -\frac{\hbar^2\nabla^2}{2m} + V(\boldsymbol{r})$

→ simultaneous diagonalization → Bloch states

$$\mathscr{H} = \frac{1}{2m}(\boldsymbol{p}+e\boldsymbol{A})^2 + V(\boldsymbol{r})$$

$\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}(\boldsymbol{r}+\boldsymbol{R}) + \boldsymbol{\nabla} g(\boldsymbol{r})$ does not have translational symmetry

Magnetic translation operator $\quad \boldsymbol{p} \to \boldsymbol{p}+e\boldsymbol{A}$

Symmetric gauge $\quad \boldsymbol{A} = \boldsymbol{B}\times\boldsymbol{r}/2$

$$T_{B\boldsymbol{R}} \equiv \exp\left\{\frac{i}{\hbar}\boldsymbol{R}\cdot\left[\boldsymbol{p}+\frac{e}{2}(\boldsymbol{r}\times\boldsymbol{B})\right]\right\} = T_{\boldsymbol{R}}\exp\left[\frac{ie}{\hbar}(\boldsymbol{B}\times\boldsymbol{R})\cdot\frac{\boldsymbol{r}}{2}\right]$$

$$[\mathscr{H}, T_{B\boldsymbol{R}}] = 0$$

Magnetic Brillouin zone

However $T_{B\boldsymbol{R}a}T_{B\boldsymbol{R}b} = \exp(2\pi i\phi)T_{B\boldsymbol{R}b}T_{B\boldsymbol{R}a}, \quad \phi = \frac{eB}{h}ab$

$\phi = p/q$: rational number

Magnetic unit cell: unit vectors $(\boldsymbol{a}, \boldsymbol{b}) \rightarrow$ magnetic unit vectors $(q\boldsymbol{a}, \boldsymbol{b})$

Lattice vector : $R' = n(q\boldsymbol{a}) + m\boldsymbol{b}$ $\quad T_{B\boldsymbol{R}'}$: elements commute

ψ: simultaneously diagonalizes $\mathcal{H}$ and $T_{B\boldsymbol{R}'}$

Magnetic Brillouin zone: $0 \le k_1 < 2\pi/qa, \; 0 \le k_2 < 2\pi/b$

$$T_{q\boldsymbol{a}+\boldsymbol{b}}\psi = \exp[i(k_x qa + k_y b)]\psi$$

Magnetic Bloch function: $\psi_{n\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\boldsymbol{r}}u_{n\boldsymbol{k}}(\boldsymbol{r})$

Magnetic Bloch function

$$u_{n\boldsymbol{k}}(x+qa,y)=\exp\left(i\frac{\pi py}{b}\right)u_{n\boldsymbol{k}}(x,y),$$

$$u_{n\boldsymbol{k}}(x,y+b)=\exp\left(-i\frac{\pi px}{qa}\right)u_{n\boldsymbol{k}}(x,y).$$

$$u_{n\boldsymbol{k}}(\boldsymbol{r})=|u_{n\boldsymbol{k}(\boldsymbol{r})}|\exp[i\theta_{\boldsymbol{k}}(\boldsymbol{r})] \quad p=-\frac{1}{2\pi}\oint d\boldsymbol{l}\cdot\frac{\partial\theta_{\boldsymbol{k}}(\boldsymbol{r})}{\partial\boldsymbol{l}}$$

Remember k·p approximation $\boldsymbol{p}e^{i\boldsymbol{k}\boldsymbol{r}}=e^{i\boldsymbol{k}\boldsymbol{r}}(\hbar\boldsymbol{k}+\boldsymbol{p})$

$$(\boldsymbol{p}+e\boldsymbol{A})^2e^{i\boldsymbol{k}\boldsymbol{r}}u_{n\boldsymbol{k}}(\boldsymbol{r})=e^{i\boldsymbol{k}\boldsymbol{r}}(\hbar\boldsymbol{k}+\boldsymbol{p}+e\boldsymbol{A})^2u_{n\boldsymbol{k}}(\boldsymbol{r})$$

Schrodinger-like equation for $u_{n\boldsymbol{k}}(\boldsymbol{r})$

$$\mathscr{H}_{\boldsymbol{k}}u_{n\boldsymbol{k}}(\boldsymbol{r})=E_{n\boldsymbol{k}}u_{n\boldsymbol{k}}(\boldsymbol{r}),\quad \mathscr{H}_{\boldsymbol{k}}=\frac{1}{2m}(-i\hbar\boldsymbol{\nabla}+\hbar\boldsymbol{k}+e\boldsymbol{A})^2+V(\boldsymbol{r})$$

k- dependent Hamiltonian

Kubo formula for σ_{xy}

Ryogo Kubo
1920 - 1995

Electric field along y-axis: E

$$|\alpha'\rangle = |\alpha\rangle + \sum_{\beta \neq \alpha} \frac{\langle \beta | eEy | \alpha \rangle}{E_\alpha - E_\beta} |\beta\rangle$$

Unperturbed state

$$j_x = \frac{1}{L^2} \sum_\alpha f(E_{\alpha'}) \langle \alpha' | \hat{j}_x | \alpha' \rangle = \frac{1}{L^2} \sum_\alpha f(E_\alpha) \sum_{\beta \neq \alpha} \frac{\langle \alpha | (-ev_x) | \beta \rangle \langle \beta | eEy | \alpha \rangle}{E_\alpha - E_\beta} + \text{c.c.}$$

$$\langle \beta | v_y | \alpha \rangle = \langle \beta | \dot{y} | \alpha \rangle = -\frac{i}{\hbar} \langle \beta | [y, \mathscr{H}] | \alpha \rangle = -\frac{i}{\hbar} (E_\alpha - E_\beta) \langle \beta | y | \alpha \rangle$$

$$\sigma_{xy} = \frac{j_x}{E} = \frac{e^2 \hbar}{i L^2} \sum_\alpha f(E_\alpha) \sum_\beta \frac{\langle \alpha | v_x | \beta \rangle \langle \beta | v_y | \alpha \rangle}{(E_\alpha - E_\beta)^2} + \text{c.c.}$$

Magnetic Bloch function (II)

Velocity operator: $\boldsymbol{v} = (-i\hbar\boldsymbol{\nabla} + e\boldsymbol{A})/m$

$$u_{n\boldsymbol{k}}(\boldsymbol{r}) \to |n, \boldsymbol{k}\rangle$$

$$\langle n, \boldsymbol{k}|\boldsymbol{v}|m, \boldsymbol{k}'\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'} \int_0^{qa} dx \int_0^{b} dy u^*_{n\boldsymbol{k}} \boldsymbol{v} u_{m\boldsymbol{k}'} \equiv \delta_{\boldsymbol{k}\boldsymbol{k}'} \langle n|\boldsymbol{v}|m\rangle$$

Normalization: $\displaystyle\int_0^{qa} dx \int_0^{b} dy |u_{n\boldsymbol{k}}(\boldsymbol{r})|^2 = 1$

$$\langle n|v_x|m\rangle = \frac{1}{\hbar}\left\langle n\left|\frac{\partial\mathscr{H}_{\boldsymbol{k}}}{\partial k_x}\right|m\right\rangle, \quad \langle n|v_y|m\rangle = \frac{1}{\hbar}\left\langle n\left|\frac{\partial\mathscr{H}_{\boldsymbol{k}}}{\partial k_y}\right|m\right\rangle.$$

$$\left\langle n\left|\frac{\partial\mathscr{H}_{\boldsymbol{k}}}{\partial k_j}\right|m\right\rangle = (E_m - E_n)\left\langle n\left|\frac{\partial u_m}{\partial k_j}\right.\right\rangle = -(E_m - E_n)\left\langle\left.\frac{\partial u_n}{\partial k_j}\right|m\right\rangle,$$

$j = x. y$

$$\sigma_{xy} = -i\frac{e^2}{\hbar}\sum_{\boldsymbol{k}}\sum_{n} f(E_{n\boldsymbol{k}}) \sum_{m(\neq n)} \left[\frac{\langle n\boldsymbol{k}|\partial\mathscr{H}_{\boldsymbol{k}}/\partial k_x|m\boldsymbol{k}\rangle\langle m\boldsymbol{k}|\partial\mathscr{H}_{\boldsymbol{k}}/\partial k_y|n\boldsymbol{k}\rangle}{(E_{n\boldsymbol{k}}-E_{m\boldsymbol{k}})^2} - \text{c.c.}\right]$$

$$= -i\frac{e^2}{\hbar}\sum_{\boldsymbol{k}}\sum_{n} f(E_{n\boldsymbol{k}}) \sum_{m(\neq n)} \left[\left\langle \frac{\partial u_n}{\partial k_x}\middle| m\right\rangle\left\langle m\middle|\frac{\partial u_n}{\partial k_y}\right\rangle - \left\langle \frac{\partial u_n}{\partial k_y}\middle| m\right\rangle\left\langle m\middle|\frac{\partial u_n}{\partial k_x}\right\rangle\right]$$

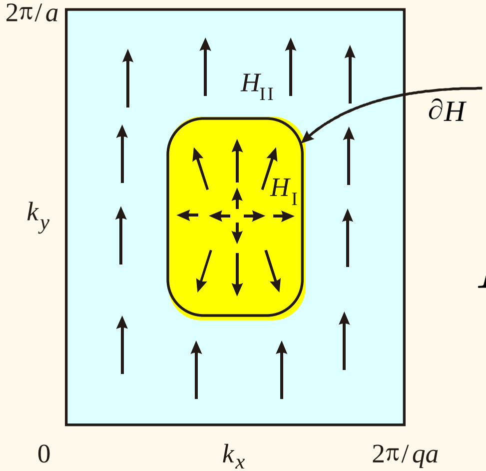
$$= \frac{e^2}{h}\frac{2\pi}{i}\sum_{\boldsymbol{k}}\sum_{n} f(E_{n\boldsymbol{k}}) \left[\left\langle \frac{\partial u_n}{\partial k_x}\middle|\frac{\partial u_n}{\partial k_y}\right\rangle - \left\langle \frac{\partial u_n}{\partial k_y}\middle|\frac{\partial u_n}{\partial k_x}\right\rangle\right].$$

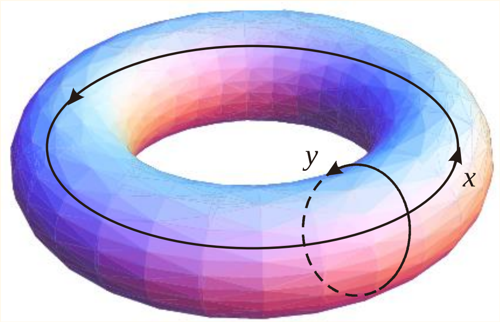
$$\text{Vector field:}\quad \boldsymbol{A}_{n\boldsymbol{k}} = \int d^2\boldsymbol{r}\, u^*_{n\boldsymbol{k}}\boldsymbol{\nabla}_{\boldsymbol{k}} u_{n\boldsymbol{k}} = \langle u_{n\boldsymbol{k}}|\boldsymbol{\nabla}_{\boldsymbol{k}}|u_{n\boldsymbol{k}}\rangle$$

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi i}\sum_{E_n<E_{\mathrm{F}}}\int_{\mathrm{MBZ}} d^2k[\boldsymbol{\nabla}_{\boldsymbol{k}}\times\boldsymbol{A}_{n\boldsymbol{k}}]_{k_z} = \frac{e^2}{h}\frac{1}{2\pi i}\sum_{E_n<E_{\mathrm{F}}}\int_{\mathrm{MBZ}} d^2k[\mathrm{rot}_{\boldsymbol{k}}\boldsymbol{A}_{n\boldsymbol{k}}]_{k_z}$$

Existence of zero or anomaly

Magnetic Brillouin zone





$$I = \frac{1}{2\pi i}\left[\int_{\mathrm{I}} d^2k[\mathrm{rot}\boldsymbol{A}]_{k_z} + \int_{\mathrm{II}} d^2k[\mathrm{rot}\boldsymbol{A}]_{k_z}\right] = \oint_{\partial H}(\boldsymbol{A}^{\mathrm{II}} - \boldsymbol{A}^{\mathrm{I}})\cdot\frac{d\boldsymbol{k}}{2\pi i}$$

On the boundary ∂H $\quad u_{\boldsymbol{k}}^{\mathrm{I}} = u_{\boldsymbol{k}}^{\mathrm{II}} e^{i\theta(\boldsymbol{k})}$

$$I = \oint_{\partial H}\left[\langle u_{\boldsymbol{k}}^{\mathrm{II}}|\boldsymbol{\nabla}_{\boldsymbol{k}}|u_{\boldsymbol{k}}^{\mathrm{II}}\rangle + (i\boldsymbol{\nabla}_{\boldsymbol{k}}\theta)\langle u_{\boldsymbol{k}}^{\mathrm{II}}|u_{\boldsymbol{k}}^{\mathrm{II}}\rangle - \langle u_{\boldsymbol{k}}^{\mathrm{II}}|\boldsymbol{\nabla}_{\boldsymbol{k}}|u_{\boldsymbol{k}}^{\mathrm{II}}\rangle\right]\cdot\frac{d\boldsymbol{k}}{2\pi i}$$

$$= \frac{\Delta_{\partial H}\theta}{2\pi} = \nu_{\mathrm{C}}$$

: Chern number (integer)

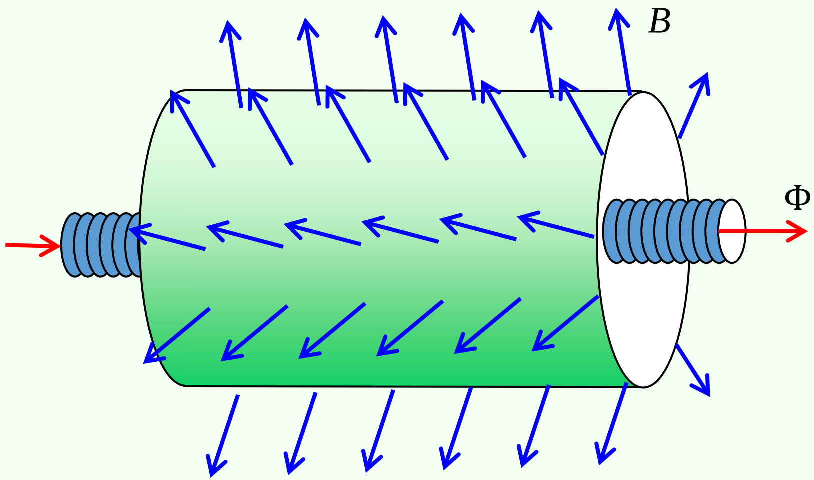
Topological invariant

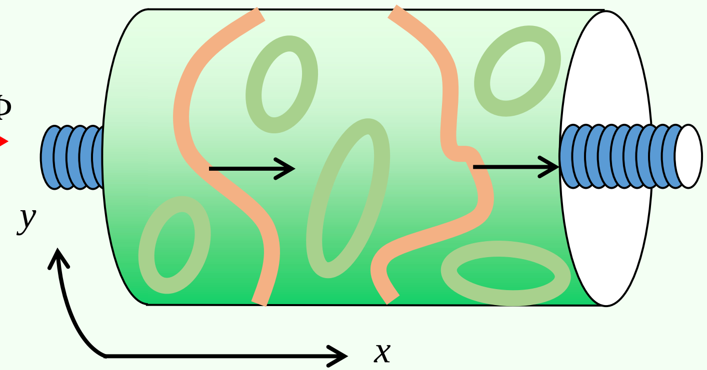
$$\sigma_{xy} = n_B \nu_{\mathrm{C}} \frac{e^2}{h}$$

Thouless-Kohmoto-Nightingale-den Nijs (TKNN) Formula

Laughlin's discussion

R. Laughlin, Phys. Rev. B **23**, 5632 (1981).





Robert Laughlin

Landau gauge $\boldsymbol{A} = (0, Bx - \Phi/L_y) = (0, B(x - \Phi/L_y B))$

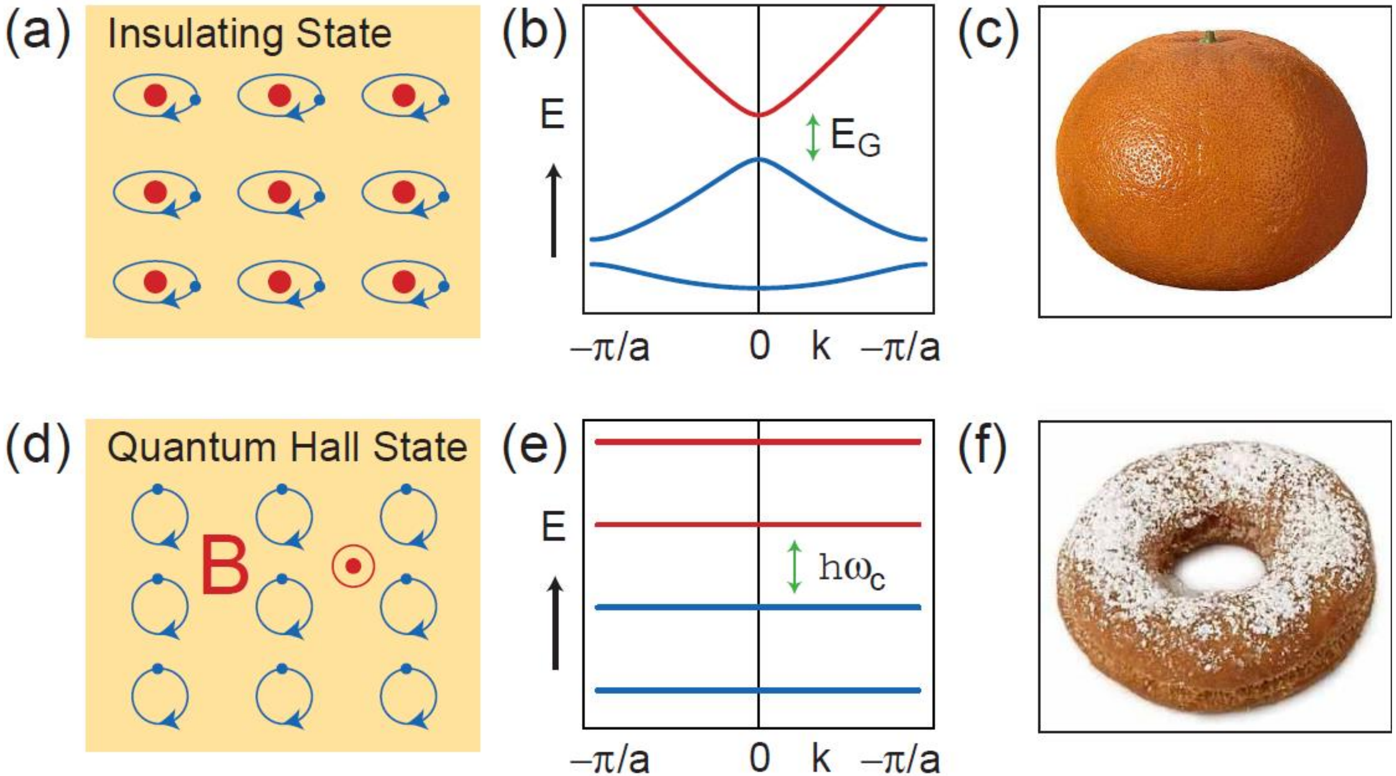
Magnetic flux Φ : X shift $X \to X + \dfrac{\Phi}{L_y B} \; : \; \dfrac{\Phi}{\phi_0}\dfrac{L_x}{N_\mathrm{L}} \quad (N_\mathrm{L} \equiv n_\mathrm{L} L_x L_y)$

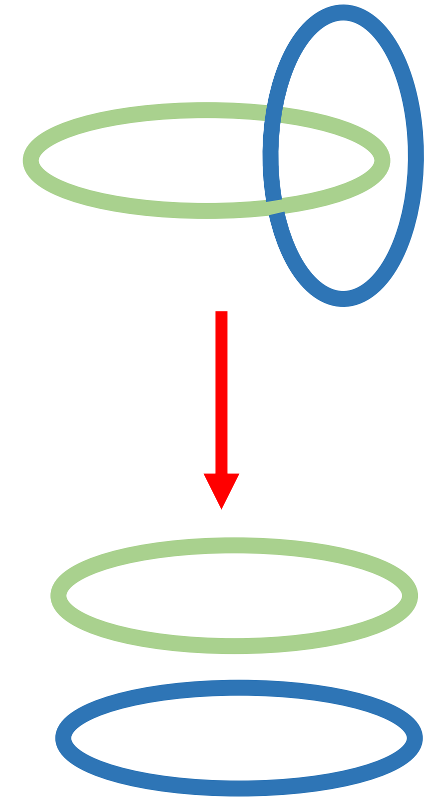
$$j_y = \frac{J_y}{L_x} = \frac{1}{L_x}\frac{\partial E_{L_x}}{\partial \Phi} \left(cf.\ E = \frac{L}{2}J^2,\ \Phi = LJ \right)$$

$$= \frac{1}{L_x}\frac{\Delta E_{L_x}}{\Delta \Phi} = \frac{1}{L_x}\left(-e\mathcal{E}_x \frac{L_x}{N_\mathrm{L}}\right)\frac{N_e}{\phi_0} = \nu \frac{e^2}{h}\mathcal{E}_x$$

Chern number =1

Bulk-Edge correspondence

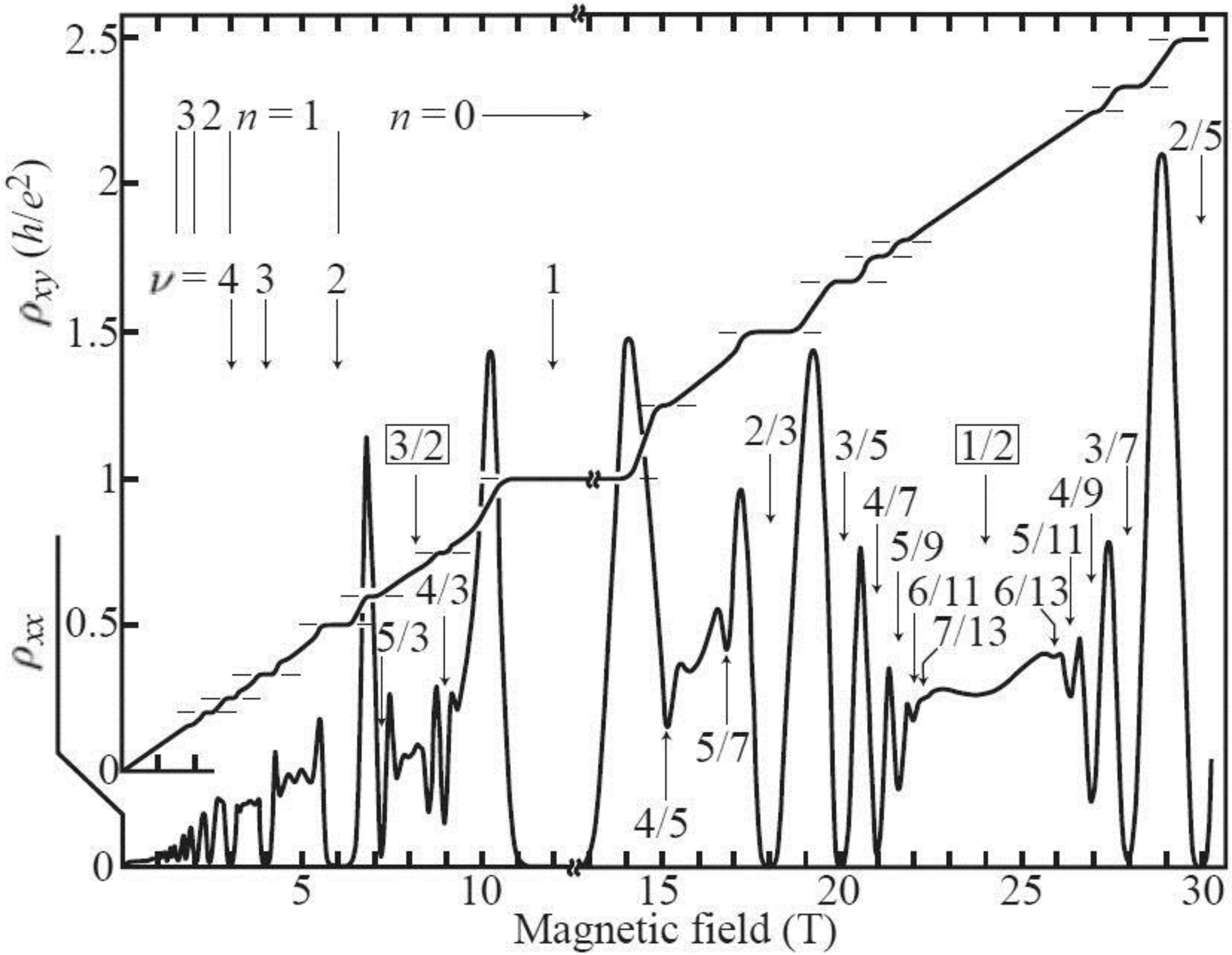




Hasan & Kane, Rev. Mod. Phys. **82**, 3045 (2010).

Transition between bands with different Chern number only can attained through energy gap collapse.

Fractional quantum Hall effect



Laughlin state

$$\psi_q(z_1, \cdots, z_{N_e}) = \prod_{i>j} (z_i - z_j)^q \exp\left(-\sum_i \frac{|z_i|^2}{4}\right)$$

Exercise E-7-4

Consider electrons in graphene under magnetic field. Let us treat the motion semi-classically.

Cyclotron frequency expression $\omega_c = \frac{eB}{m}$ cannot be applied due to m=0.

Instead use the relation $E = pc$.

(1) Express the cyclotron radius with momentum and e and B.

(2) A circular motion of electron gives kinetic phase and AB phase to the electron. Replace the momentum in (1) with $\hbar k$ and express the total phase acquired by the electron within a circular motion with the flux through the cyclotron circle and the flux quantum.

(3) The acquired phase should be integer times 2π. Obtain the expression for cyclotron radius and energy.