Lecture on

Semiconductors / 半導体

(Physics of semiconductors)

2021.7.7 Lecture 13

10:25 – 11:55

Institute for Solid State Physics, University of Tokyo

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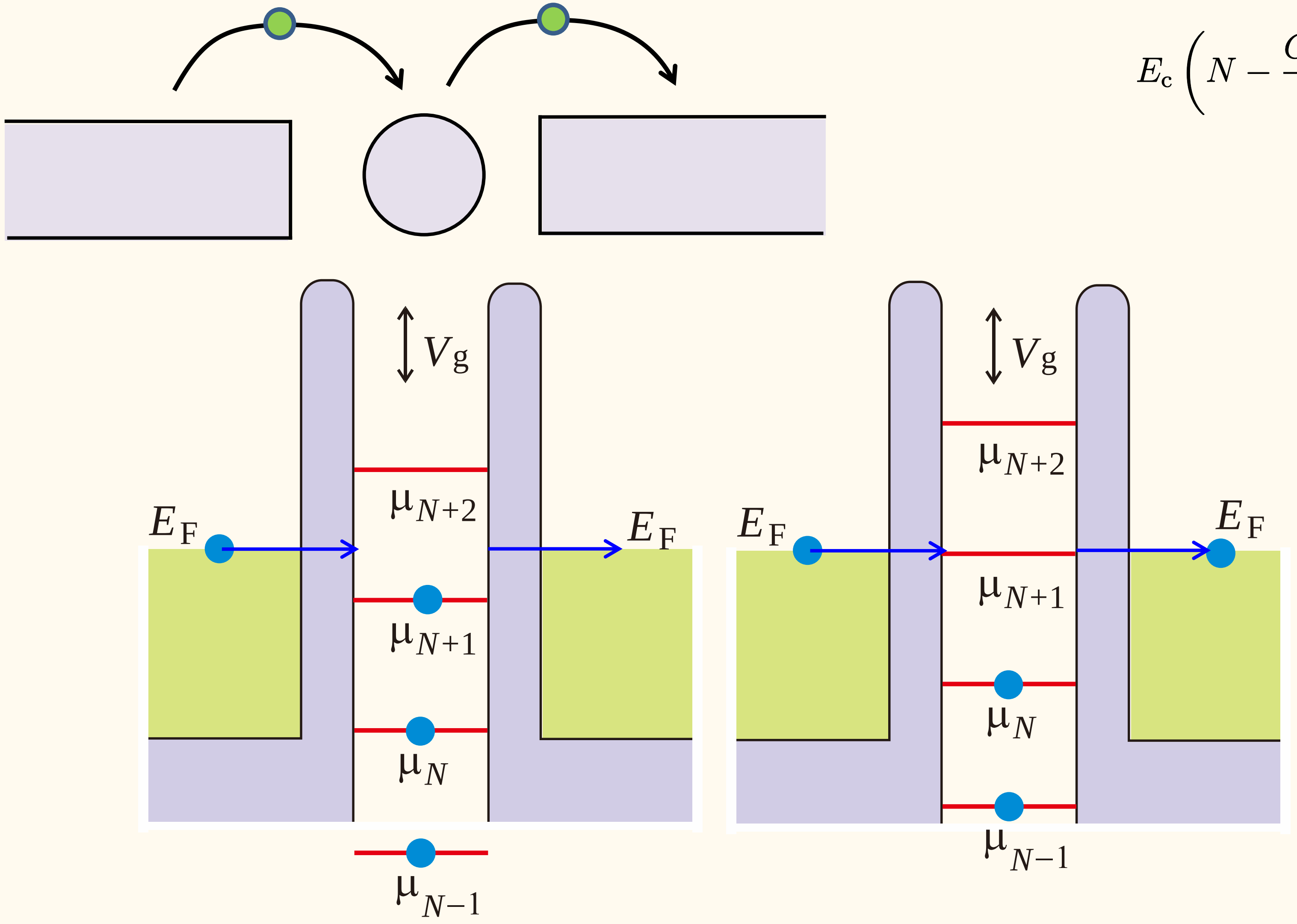
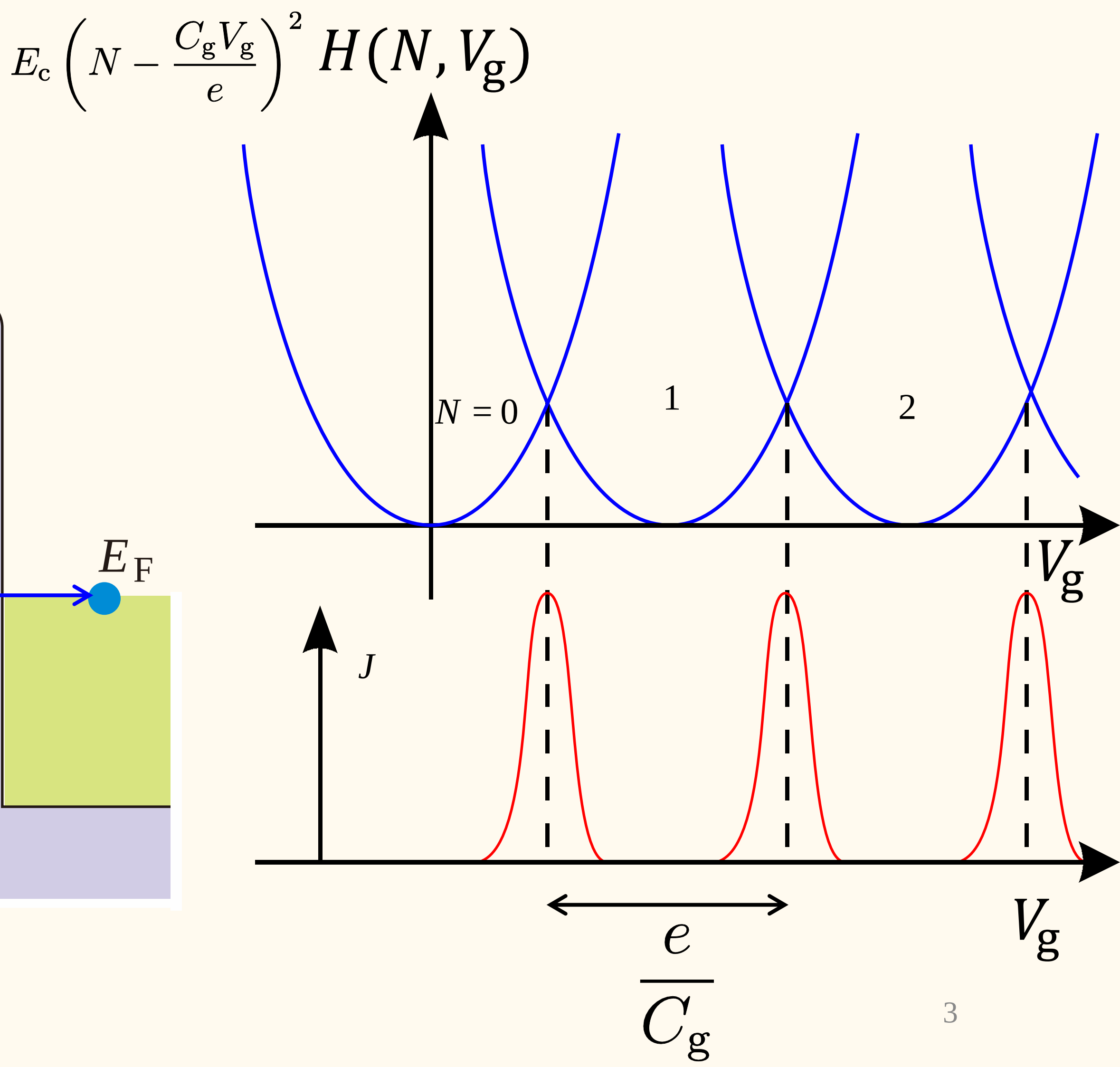
Review of the lecture in the last week

- Aharonov-Bohm effect and quantum transport
- Bunching and anti-bunching of particles (bosons and fermions)
- Waveguide propagation of exciton-polaritons
- Bose-Einstein condensation of exciton-polaritons
- Single electron effect in quantum dots

Review: Single electron effect in transport through quantum dots

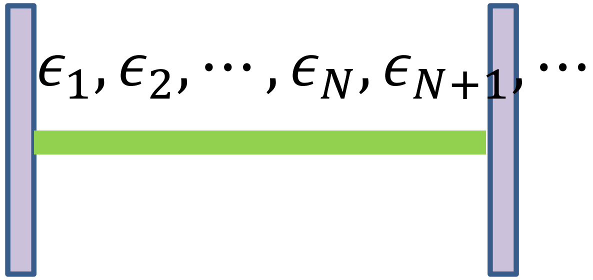
Enthalpy $H(N, V_g) = \frac{(Ne - C_g V_g)^2}{2(C + C_g)} \equiv \frac{(Ne - C_g V_g)^2}{2C_s}$

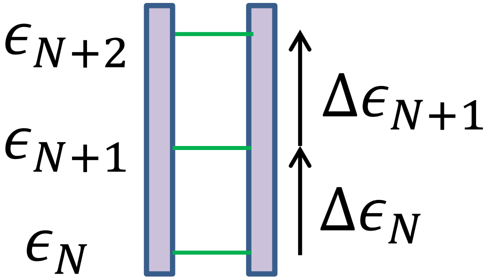


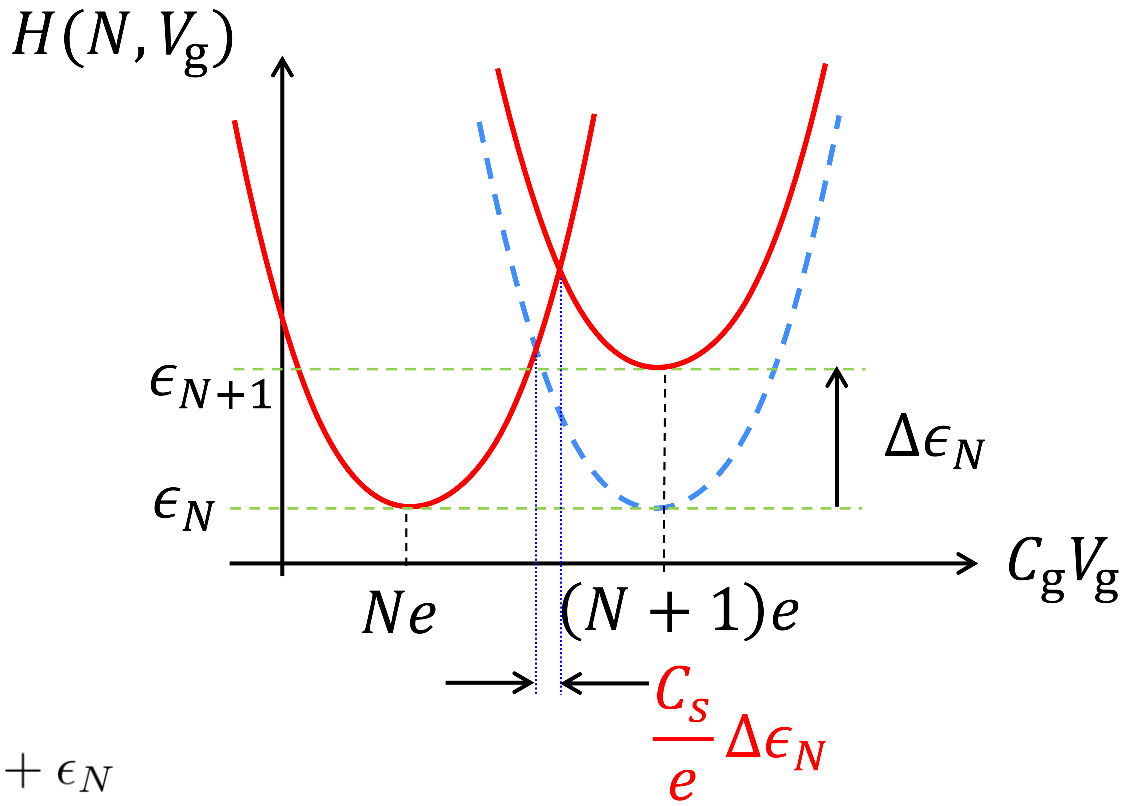


Quantum confinement

Zero-dimensional confinement to a quantum dot gives shifts in Coulomb peak positions.







Enthalpy shift by quantum confinement

$$H(N) = \frac{(Ne - C_g V_g)^2}{2C_s} + \epsilon_N$$

Chemical potential shift

$$\Delta H(N, N+1) = H(N+1) - H(N)$$

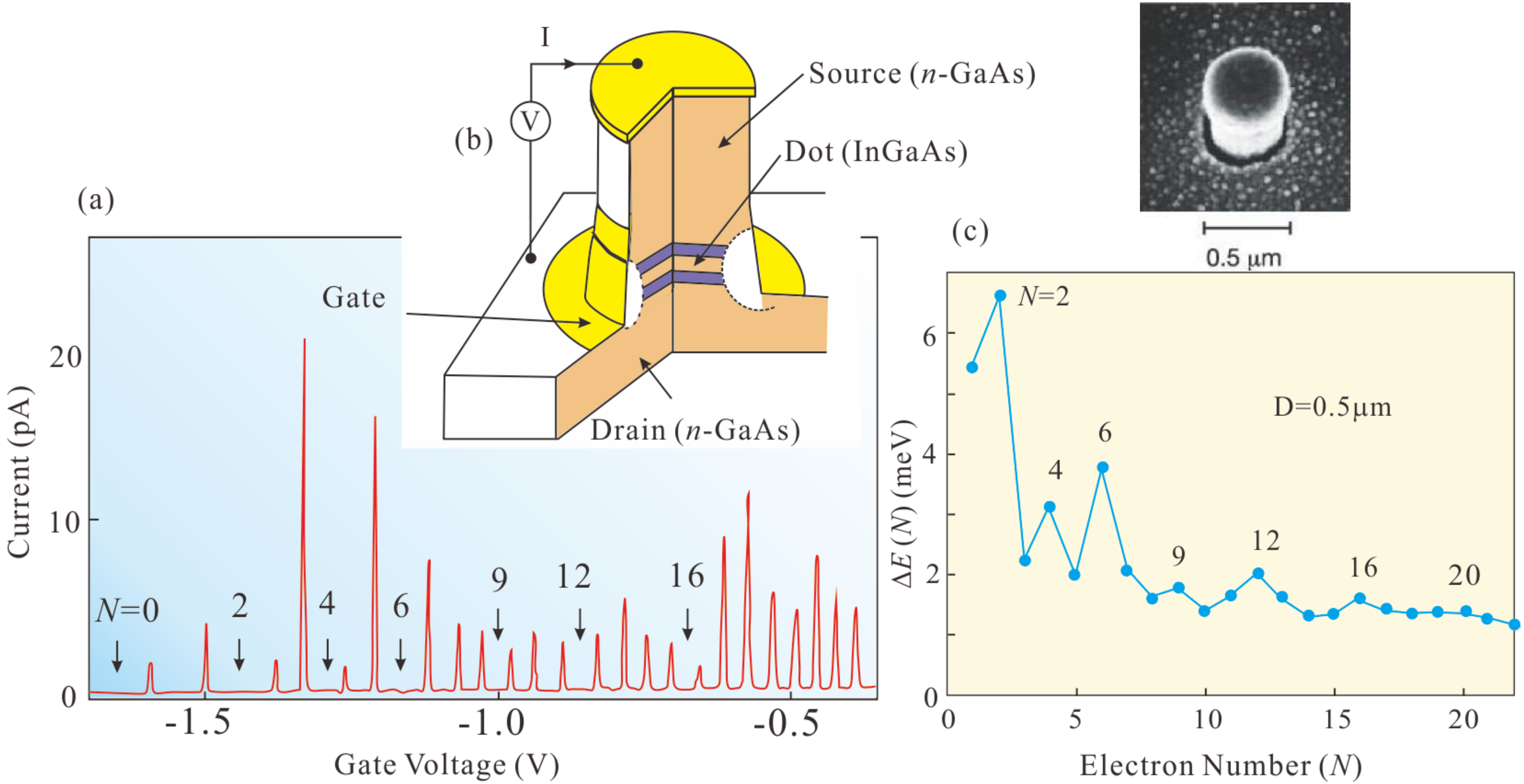
$$= \frac{e}{C_s}\left\{\left(N + \frac{1}{2}\right)e - C_g V_g\right\} + \Delta\epsilon_N$$

$$\Delta\epsilon_N \equiv \epsilon_{N+1} - \epsilon_N$$

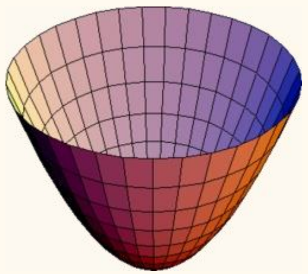
Shift in gate voltage

$$V_{gX}(N, N+1) = \frac{1}{C_g}\left\{\left(N + \frac{1}{2}\right)e + \frac{C_s}{e}\Delta\epsilon_N\right\}$$

Quantum confinement effect in a vertical quantum dot



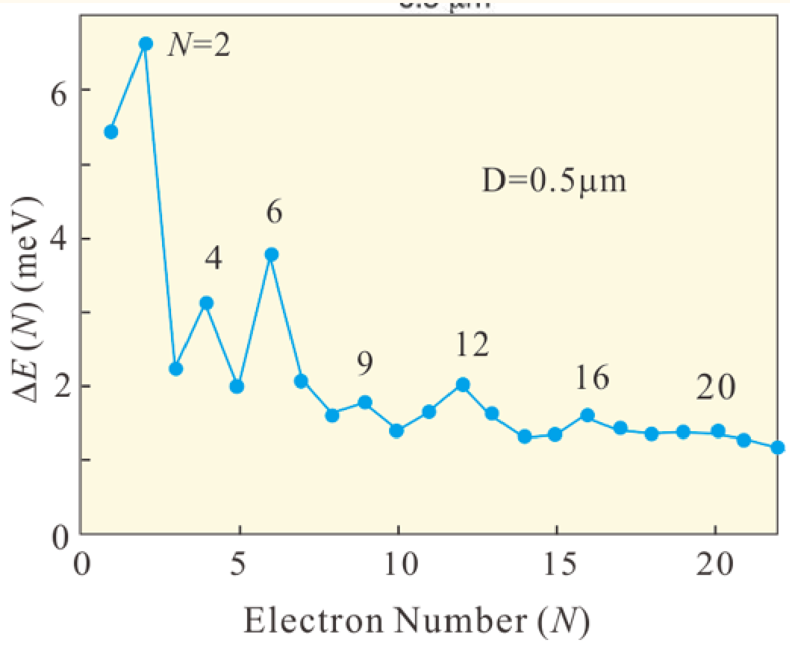
Two-dimensional harmonic potential

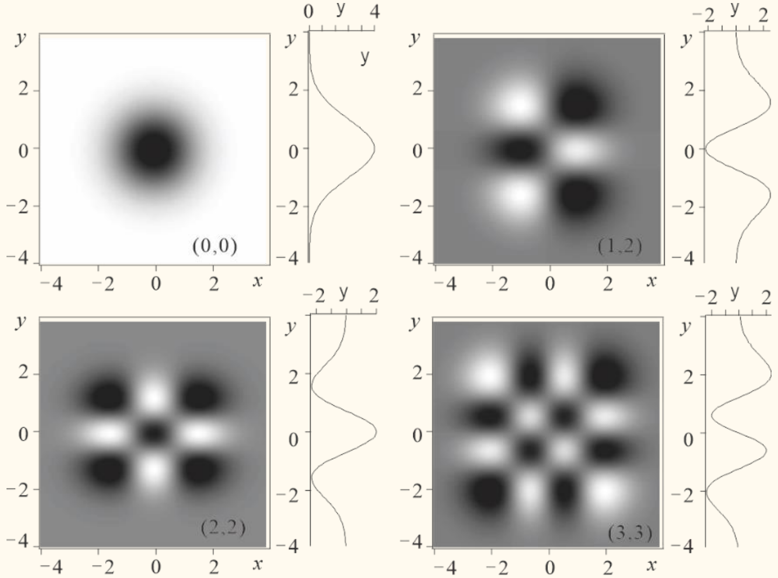


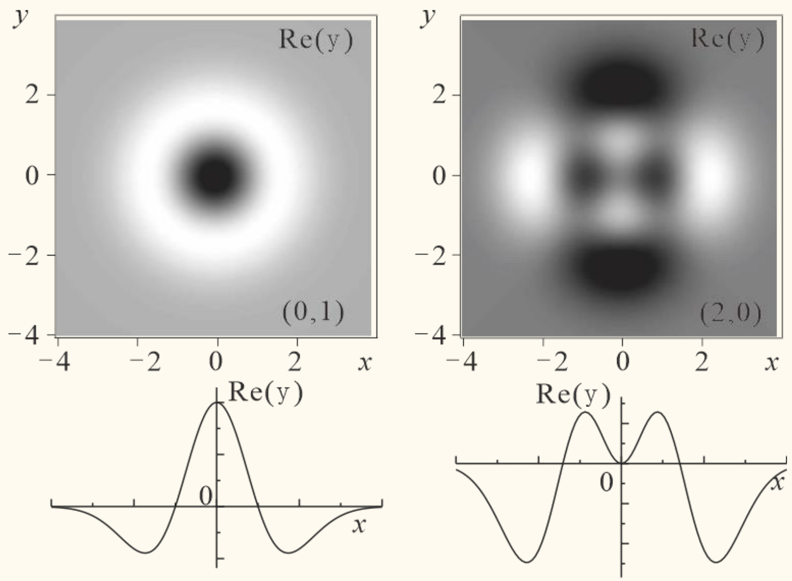
Potential shape: $V(x,y) = \frac{m\omega}{2}(x^2+y^2)$

Easy solutions from 1d harmonic potential $\psi_{n_x n_y} = A \exp\left[-\frac{m\omega(x^2+y^2)}{2\hbar}\right] H_{n_x}\left[\sqrt{\frac{m\omega}{\hbar}}x\right] H_{n_y}\left[\sqrt{\frac{m\omega}{\hbar}}y\right]$

Eigen energies: $E(n_x,n_y) = (n_x+n_y+1)\hbar\omega = (n_t+1)\hbar\omega \quad n_x+n_y \equiv n_t = 0,1,2,\cdots$







For fixed n_t $\quad n_x = 0,1,2,\cdots,n_t$ $\qquad n_t + 1$ degeneracy

With spin degeneracy: 2, 4, 6, 8, ⋯

$N = 2, 6, 12, 20, \cdots, (n+1)(n+2), \cdots$

Quantum dot in magnetic field

Hamiltonian with $\boldsymbol{B} = (0,0,B)$

$$\mathscr{H} = \frac{(\boldsymbol{p} + e\boldsymbol{A})^2}{2m} + \frac{m}{2}\omega^2(x^2 + y^2) \quad \boldsymbol{A} = \left(-\frac{By}{2}, \frac{Bx}{2}, 0\right)$$

Expansion of the kinetic energy term

$$\frac{(\boldsymbol{p} + e\boldsymbol{A})^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{ie\hbar B}{2m}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{e^2B^2}{8m}(x^2 + y^2)$$

Definition of cyclotron frequency and composite harmonic confinement potential frequency

$$\omega_c = \frac{eB}{m} \quad \Omega \equiv \sqrt{\omega^2 + (\omega_c/2)^2}$$

The Hamiltonian is rewritten as

$$\mathscr{H} = \frac{\hbar^2\nabla^2}{2m} + \frac{m}{2}\Omega^2(x^2 + y^2) + \frac{\omega_c \hat{L}_z}{2} = \mathscr{H}_\Omega + \frac{\omega_c \hat{L}_z}{2}$$

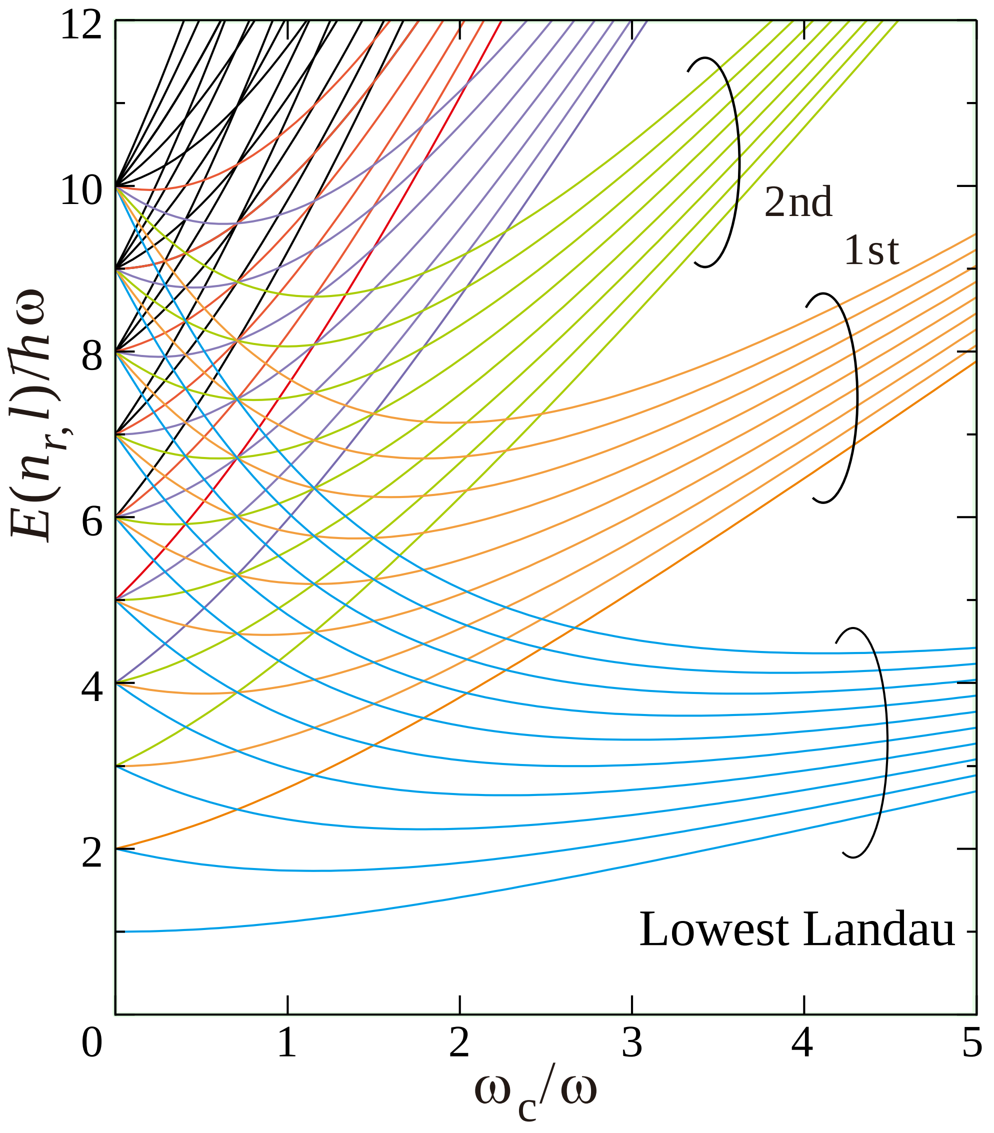
Fock-Darwin state eigen energies

$$E(n_r, l) = \hbar\Omega(2n_r + |l| + 1) + \hbar\omega_c l/2$$

Degree of degeneracy at $B = 0$

$$2n_r + |l| + 1$$

Quantum dot in magnetic field



$$\mathscr{H} = \frac{(\boldsymbol{p}+e\boldsymbol{A})^2}{2m} + \frac{m}{2}\omega^2(x^2+y^2) \quad \boldsymbol{A} = \left(-\frac{By}{2}, \frac{Bx}{2}, 0\right)$$

$$\frac{(\boldsymbol{p}+e\boldsymbol{A})^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - \frac{ie\hbar B}{2m}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{e^2B^2}{8m}(x^2+y^2)$$

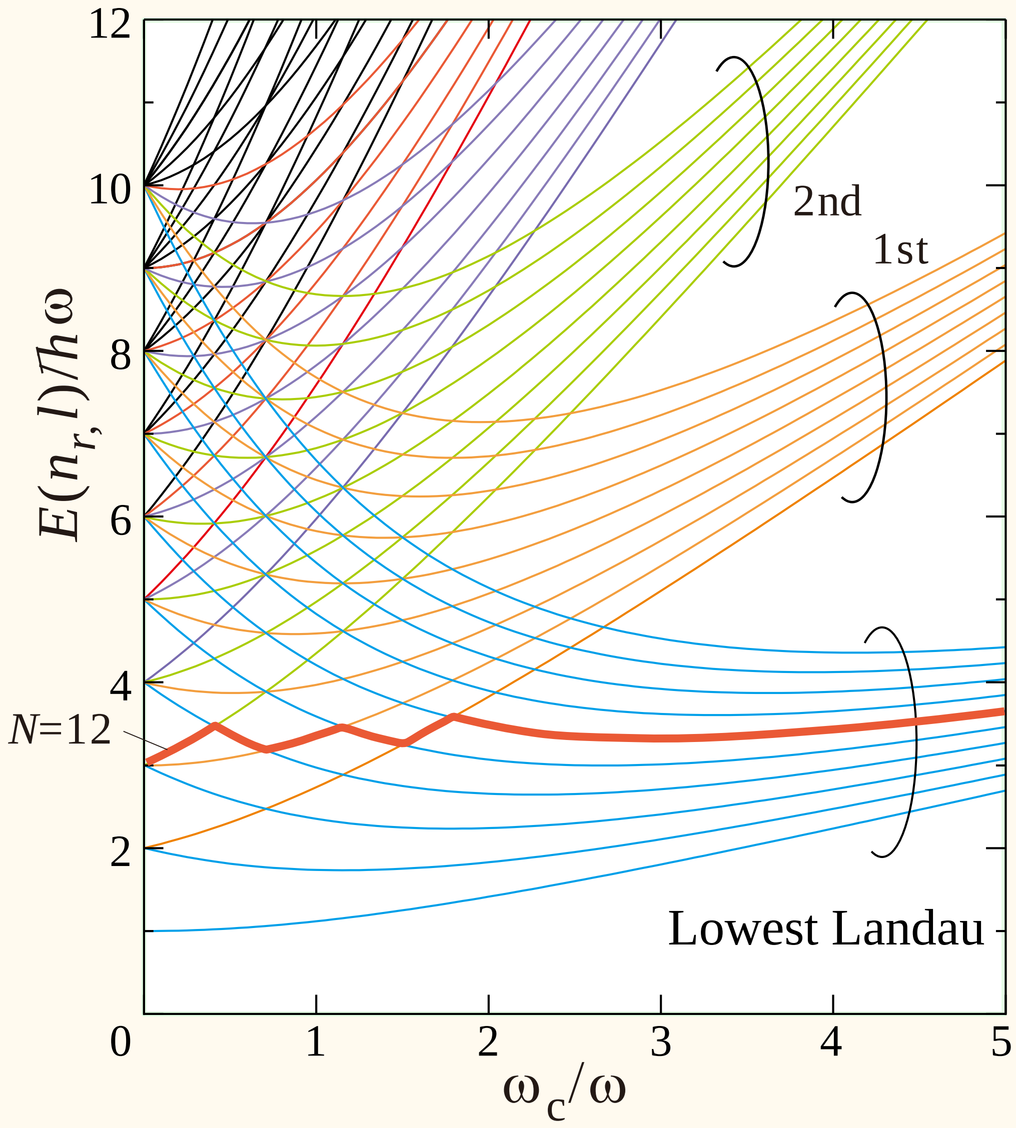
$$\omega_c = \frac{eB}{m} \quad \Omega \equiv \sqrt{\omega^2 + (\omega_c/2)^2}$$

$$\mathscr{H} = \frac{\hbar^2\nabla^2}{2m} + \frac{m}{2}\Omega^2(x^2+y^2) + \frac{\omega_c\hat{L}_z}{2} = \mathscr{H}_\Omega + \frac{\omega_c\hat{L}_z}{2}$$

$$E(n_r,l) = \hbar\Omega(2n_r+|l|+1) + \hbar\omega_c l/2$$

$$2n_r + |l| + 1$$

Fock-Darwin states



Level crossing points

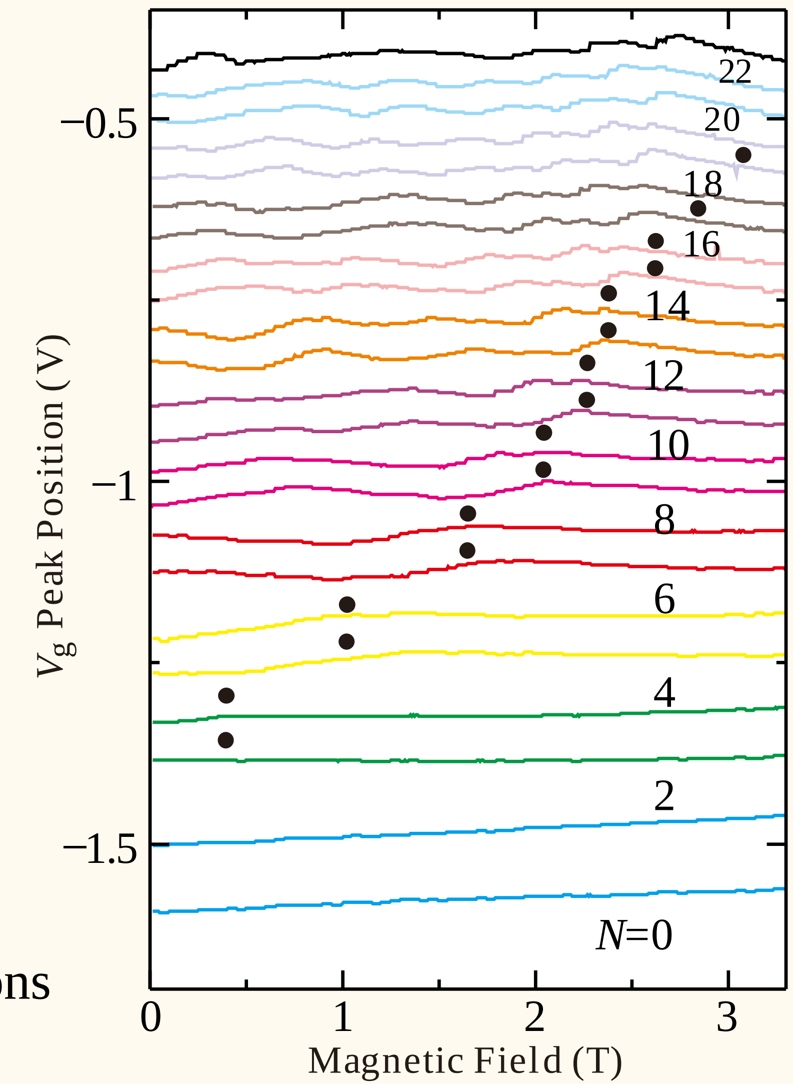
$$\left(\frac{\omega_c}{\omega}\right)^2 = n_L - 2 + \frac{1}{n_L}$$

n_L: Landau index

$= 1, 2, \ldots.$

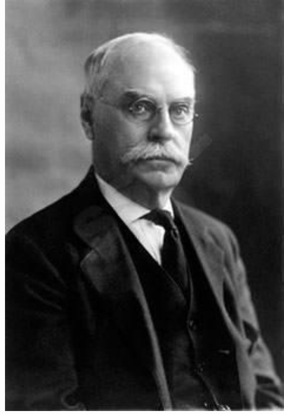
$= n_r + (|l| + l)/2$

• : Solutions

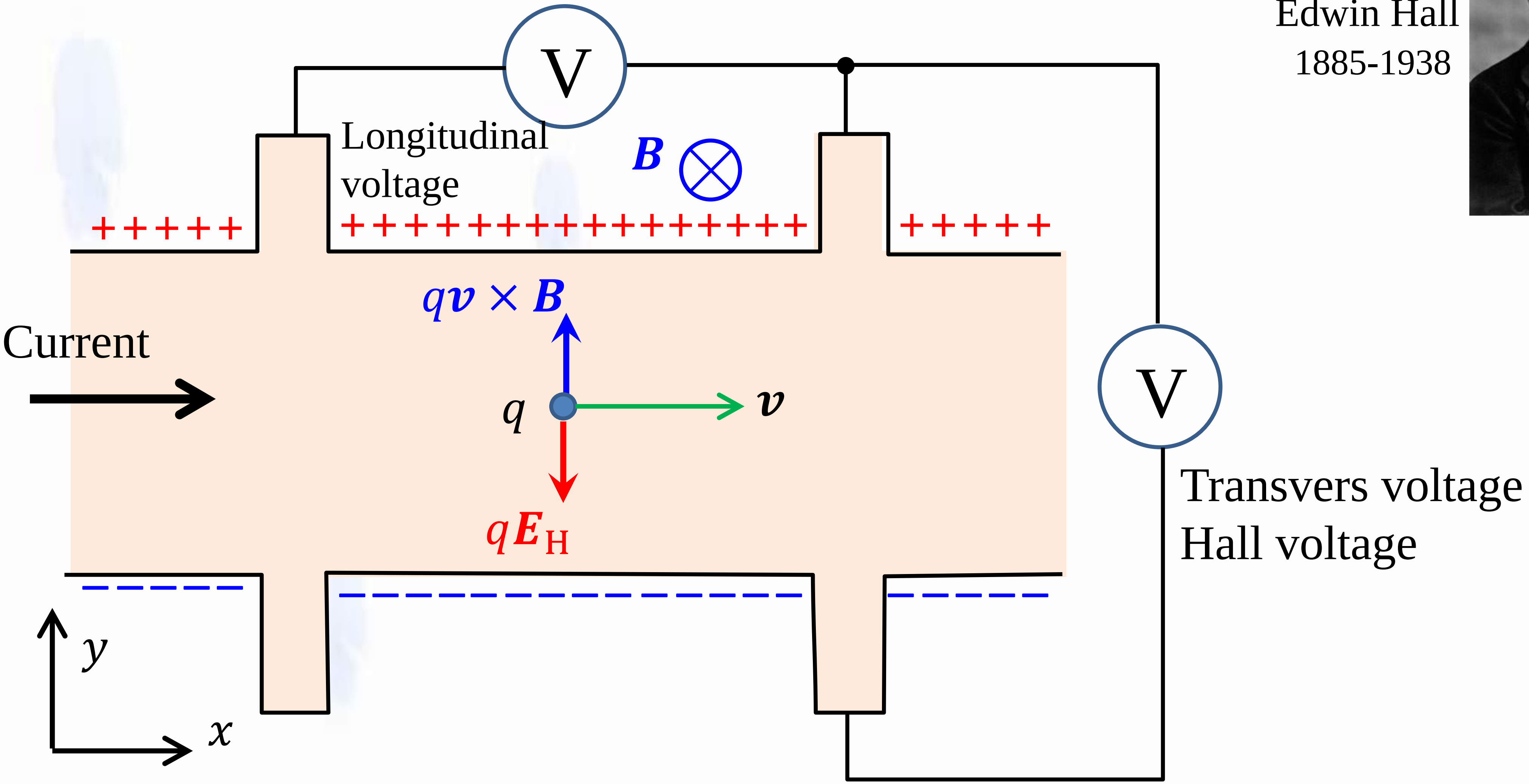


Chapter 9 Quantum Hall effect

Review: the Hall effect



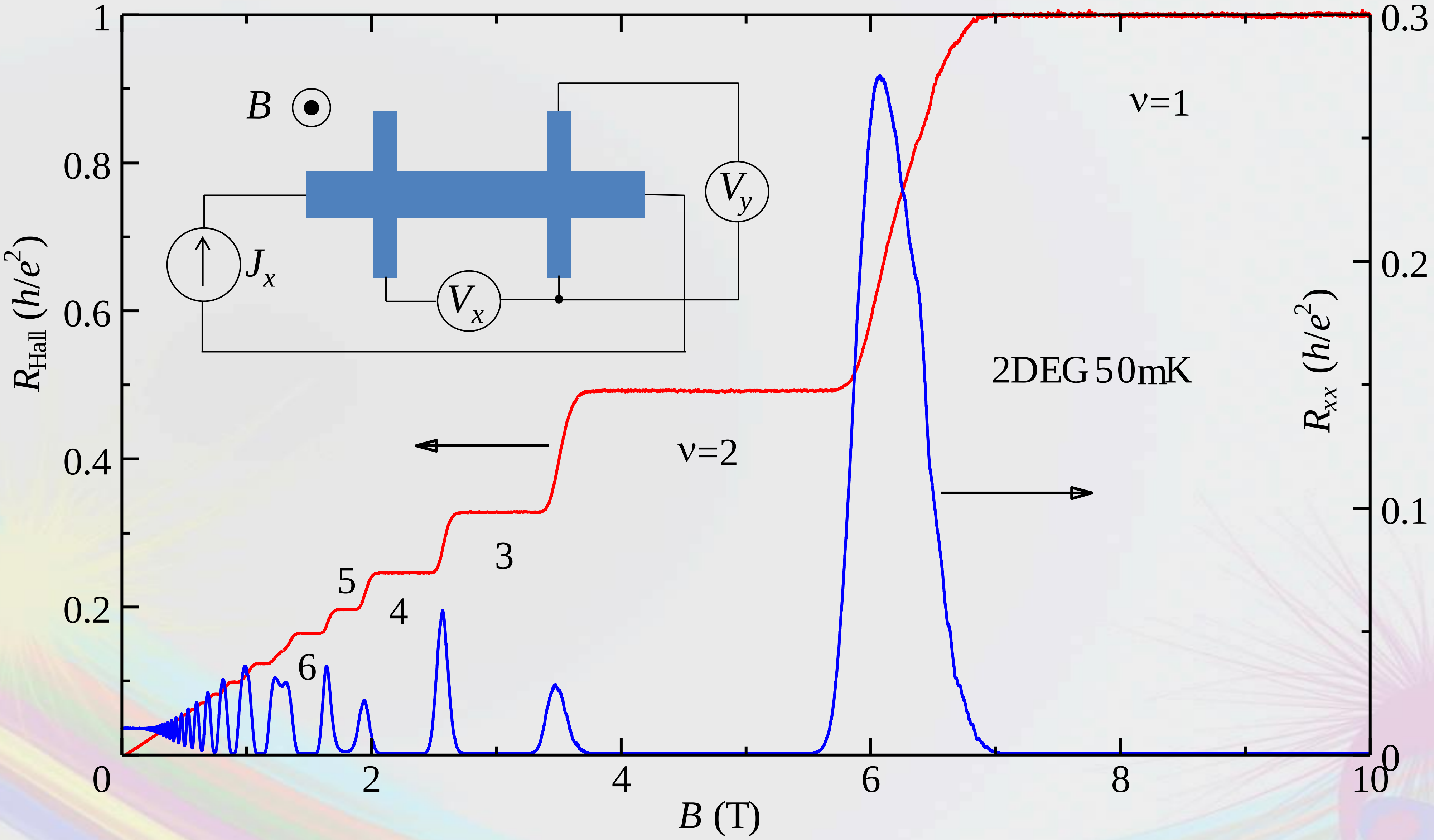
Edwin Hall
1885-1938



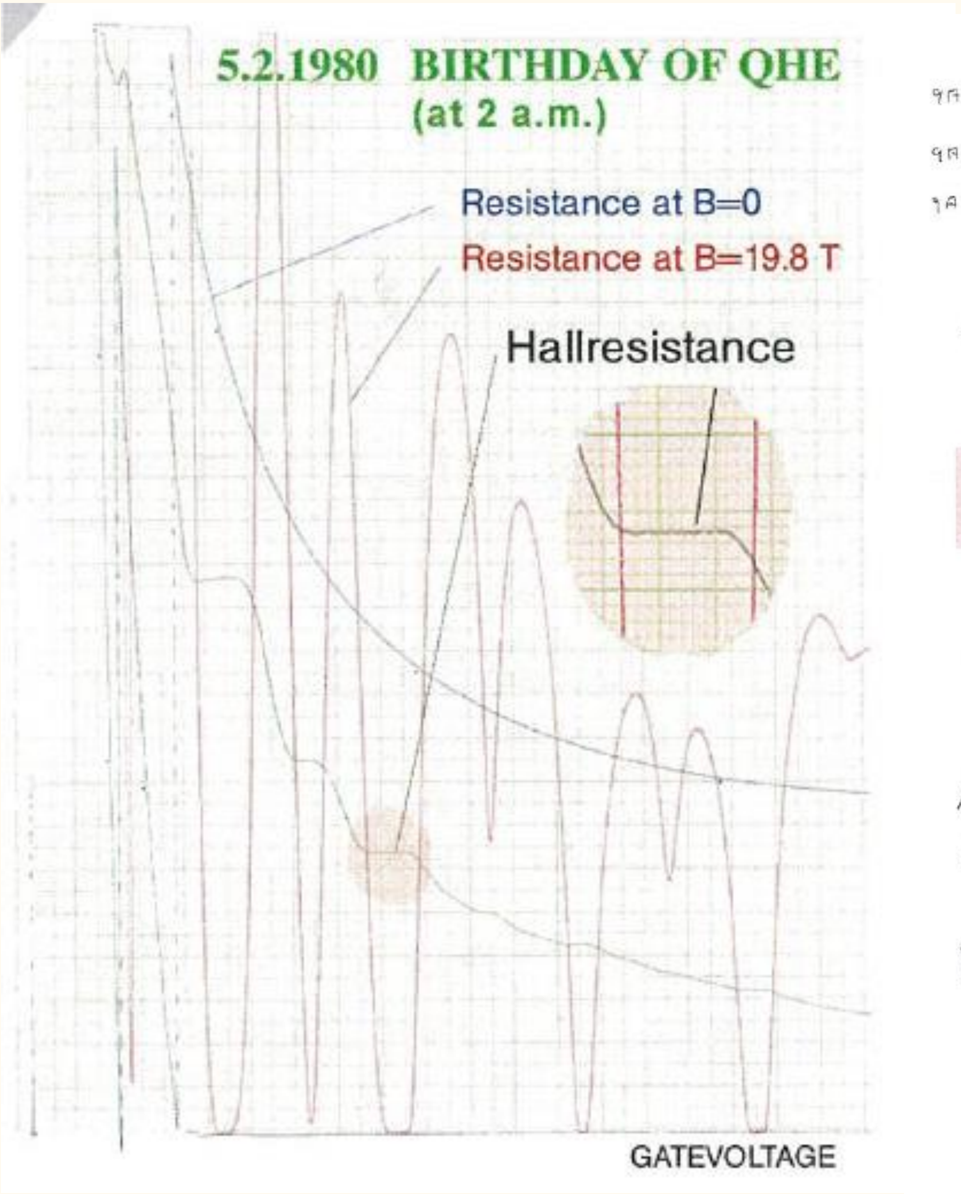
$$R_{xx} = \frac{\text{(longitudinal voltage)}}{\text{(current)}}$$ (longitudinal resistance)

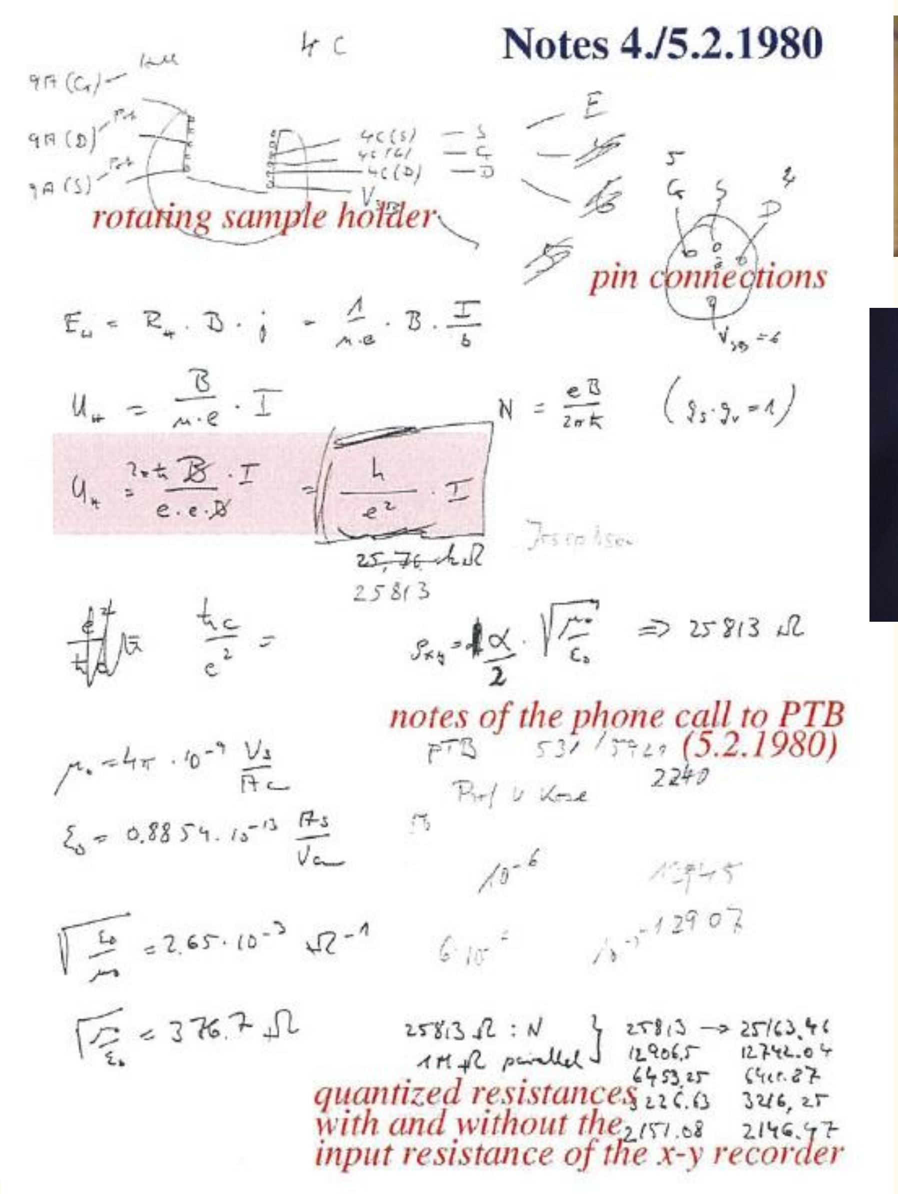
$$R_{xy} = \frac{\text{(Hall voltage)}}{\text{(current)}}$$ (Hall resistance)

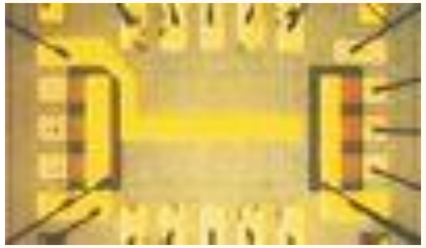
Integer Quantum Hall Effect



Birthday of quantum Hall effect

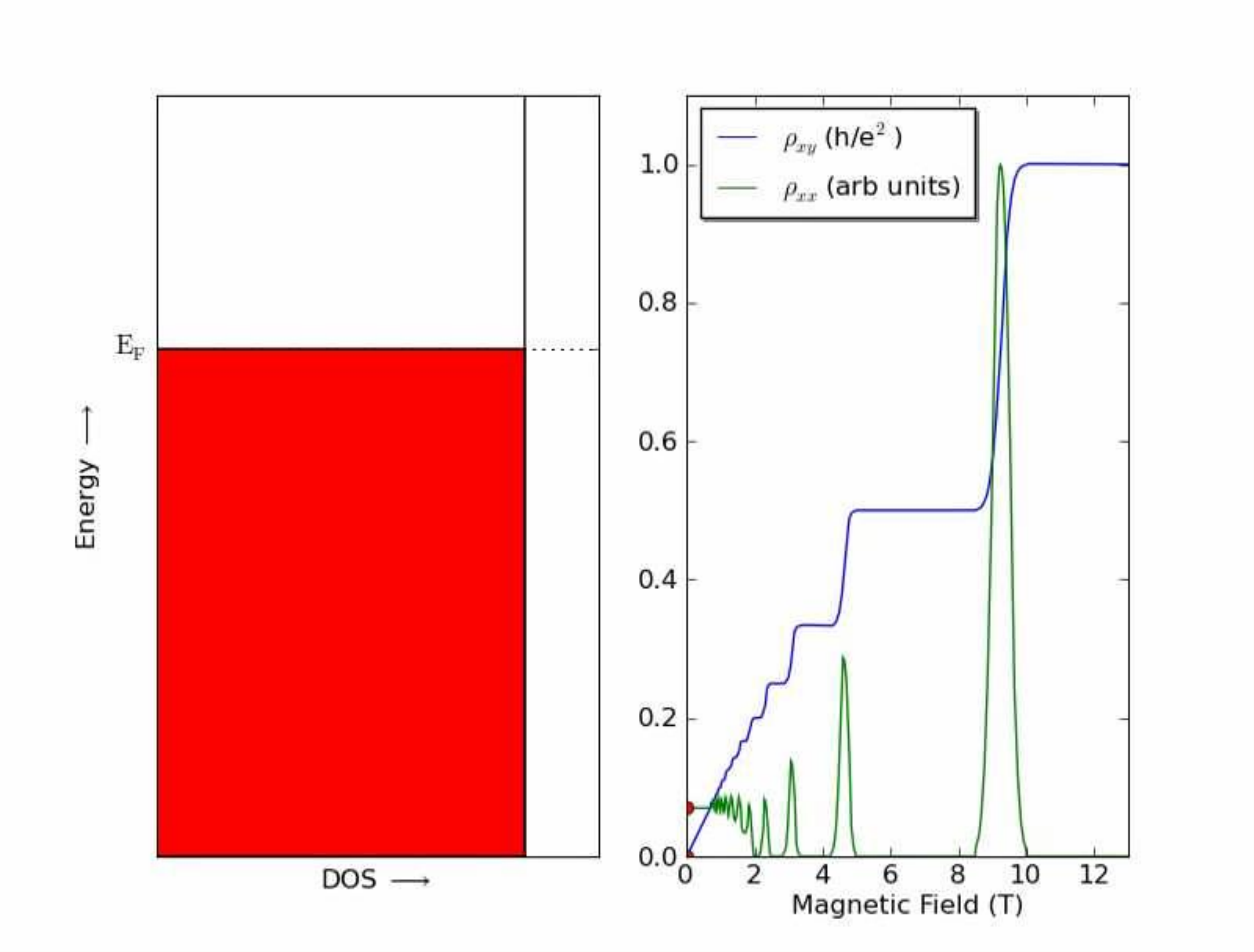






Klaus von Klitzing

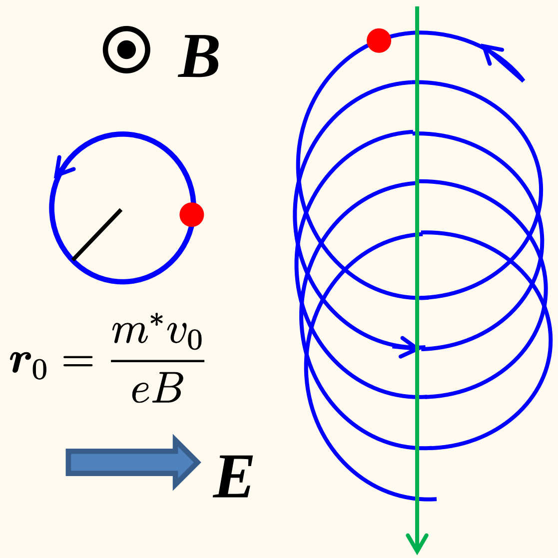
IQHE and Landau quantization



From Wikipedia

Two dimensional electrons under magnetic field

Hendrik Lorentz
1853 - 1928



Lorentz force (magnetic field only) $m\frac{d^2\boldsymbol{r}}{dt^2} = -e\boldsymbol{v}\times\boldsymbol{B}$

Cyclotron motion $\boldsymbol{r} = \boldsymbol{R} + r_0(\cos\omega_c t, \sin\omega_c t)$

$\omega_c \equiv \frac{eB}{m}$: cyclotron frequency, $r_0 \equiv \frac{v_0}{\omega_c}$: cyclotron radius,

$\boldsymbol{R}$: guiding center

This can be viewed as a motion in harmonic potential.

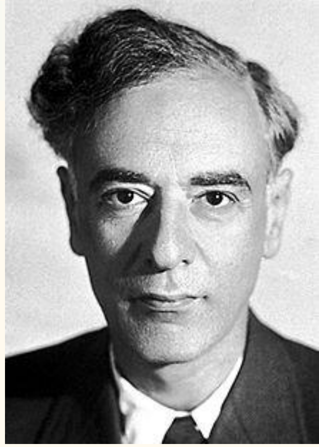
With electric field $m\frac{d^2\boldsymbol{r}}{dt^2} = -e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$

$\boldsymbol{R}$: Moves vertically to $\boldsymbol{E}$ with constant velocity E/B

Quantum mechanical Hamiltonian (no external electric field)

$$\mathscr{H} = \frac{m}{2}\boldsymbol{v}^2 = \frac{(\boldsymbol{p}+e\boldsymbol{A})^2}{2m} \equiv \frac{\boldsymbol{\pi}^2}{2m} = \frac{\pi_x^2+\pi_y^2}{2m} \qquad \boldsymbol{\pi} \equiv \boldsymbol{p} + e\boldsymbol{A}$$

Landau quantization (two-dimensional)



Lev Landau
1908 - 1968

Commutation relation $[\pi_\alpha, \beta] = -i\hbar\delta_{\alpha\beta}\ (\alpha, \beta = x, y), \quad [\pi_x, \pi_y] = -i\dfrac{\hbar^2}{l^2}$

Magnetic length $l \equiv \sqrt{\dfrac{\hbar}{eB}} = \sqrt{\dfrac{1}{2}}\sqrt{\dfrac{\phi_0}{\pi B}} \qquad (2\pi l^2)B = \phi_0 = \dfrac{h}{e}$

Space coordinate operator $\hat{\boldsymbol{r}} = \hat{\boldsymbol{R}} + \dfrac{l^2}{\hbar}(\pi_y, -\pi_x)$

Guiding center operator $\hat{\boldsymbol{R}} = (\hat{X}, \hat{Y}), \quad [\hat{X}, \hat{Y}] = il^2$

down/up operator $a = \dfrac{l}{\sqrt{2}\hbar}(\pi_x - i\pi_y), \quad a^\dagger = \dfrac{l}{\sqrt{2}\hbar}(\pi_x + i\pi_y)$

Remember:

1-d harmonic oscillator $\dfrac{\hbar\omega}{2}\left(-\dfrac{d^2}{dq^2} + q^2\right)\phi = E\phi$ down/up operators $a, a^\dagger = \dfrac{1}{\sqrt{2}}\left(\pm\dfrac{d}{dq} + q\right), \quad [a, a^\dagger] = 1$

$$[a, a^\dagger] = 1, \quad \mathscr{H} = \hbar\omega_c\left(a^\dagger a + \frac{1}{2}\right) \quad E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)\ (n = 0, 1, 2, \cdots)$$

Landau quantization: Landau gauge

Diagonalize X : Landau gauge $\boldsymbol{A} = (0, Bx)$

Schrödinger equation $$\mathscr{H}\psi = \frac{(\boldsymbol{p}+e\boldsymbol{A})^2}{2m}\psi = -\frac{1}{2m}\left[\frac{\hbar^2\partial^2}{\partial x^2} - \left(-i\hbar\frac{\partial}{\partial y} + eBx\right)^2\right]\psi(\boldsymbol{r})$$

$$= \frac{1}{2m}\left[-\hbar^2\nabla^2 - 2i\hbar eBx\frac{\partial}{\partial y} + e^2B^2x^2\right]\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r})$$

Plane wave solution along *y* $\psi(\boldsymbol{r}) = u(x)\exp(iky)$

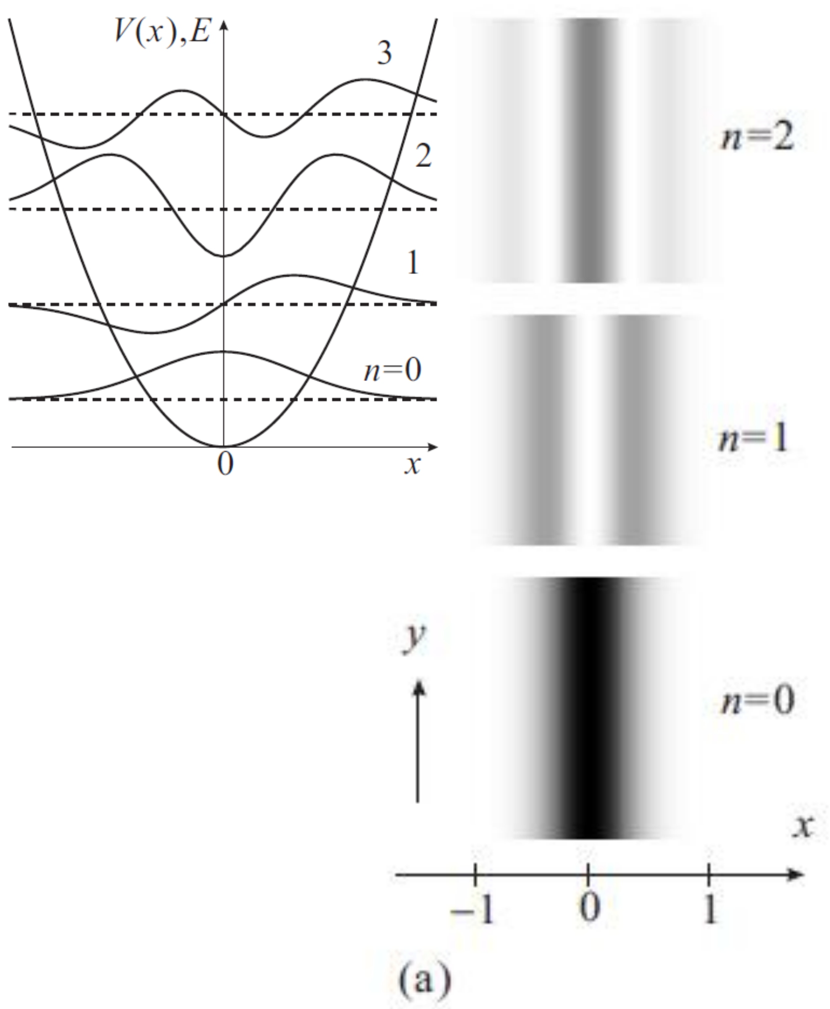
$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{(eB)^2}{2m}\left(x + \frac{\hbar}{eB}k\right)^2\right]u(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega_c^2}{2}(x + l^2k)^2\right]u(x) = Eu(x)$$

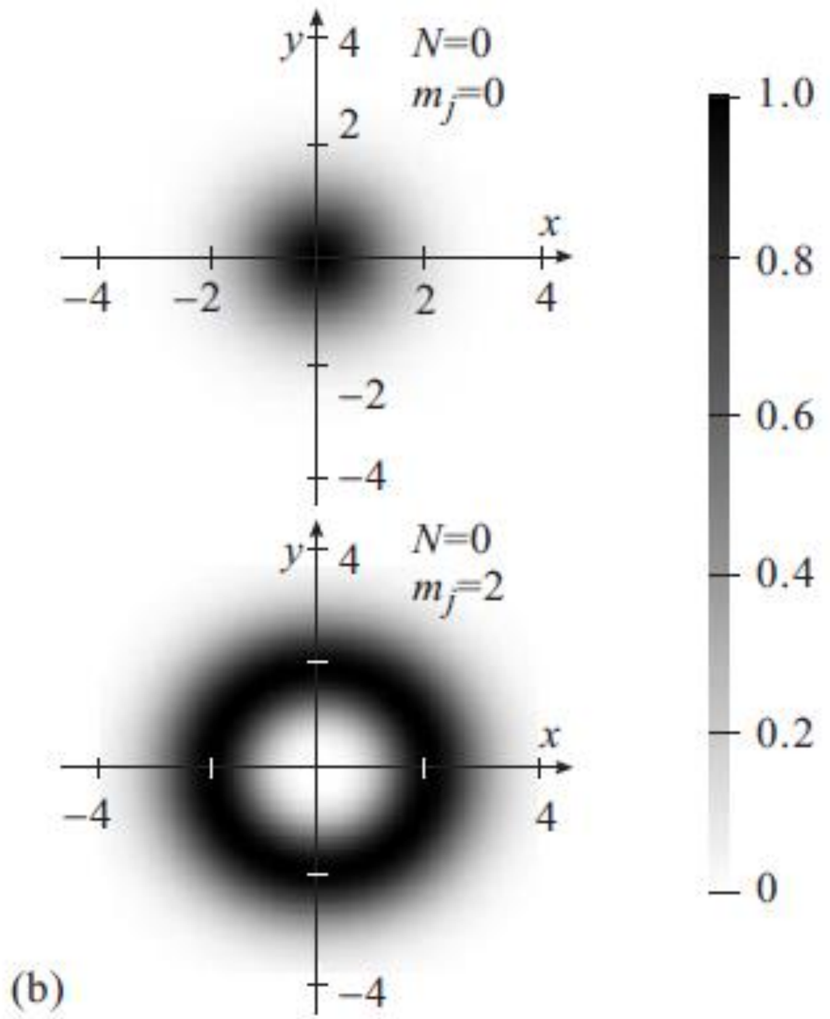
Harmonic oscillator solution $\psi_{nk}(\boldsymbol{r}) \propto H_n\left(\frac{x - x_k}{l}\right)\exp\left(-\frac{(x-x_k)^2}{2l^2}\right)\exp(iky) \quad (x_k \equiv -l^2k)$

x-direction Gaussian center $X = x_k = -l^2k = -l^2p_y/\hbar$

y-direction group velocity = 0 $\frac{dE}{dk} = 0$

Landau quantization: forms of wavefunctions





Diagonalize X

Diagonalize $X^2 + Y^2$

← Symmetric gauge $\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{r}/2$

Shubnikov-de Haas oscillation

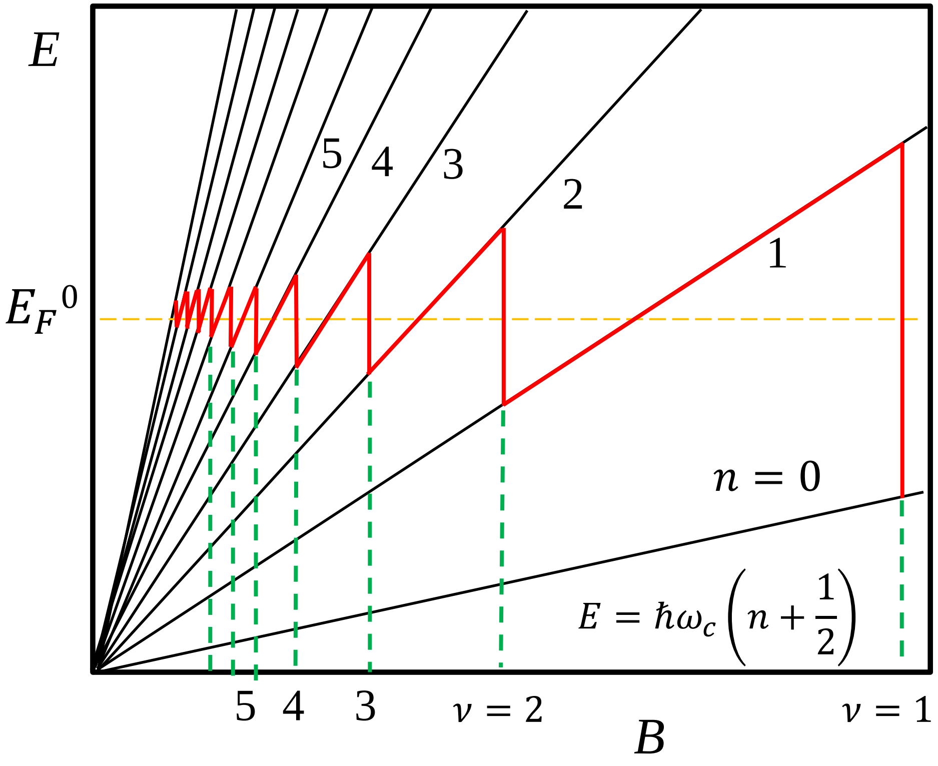
Number of states in $S = W_x \times W_y$

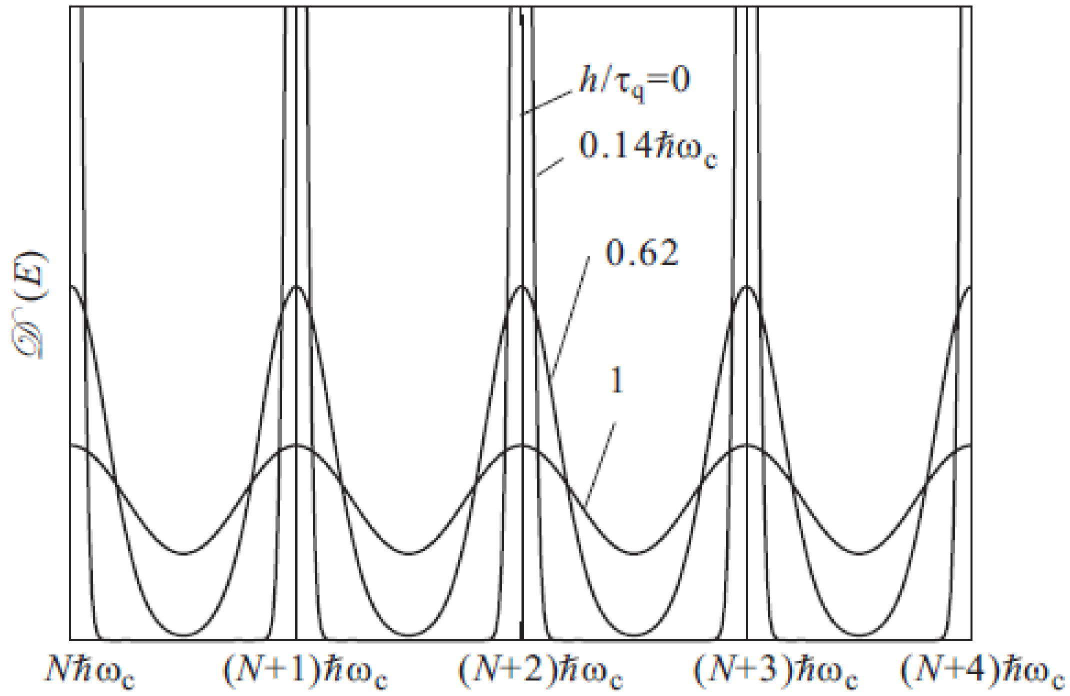
$$0 \leq X \leq W_x \rightarrow -W_x l^2 \leq k \leq 0$$

"Distance" of k-values in y-direction: $2\pi/W_y$

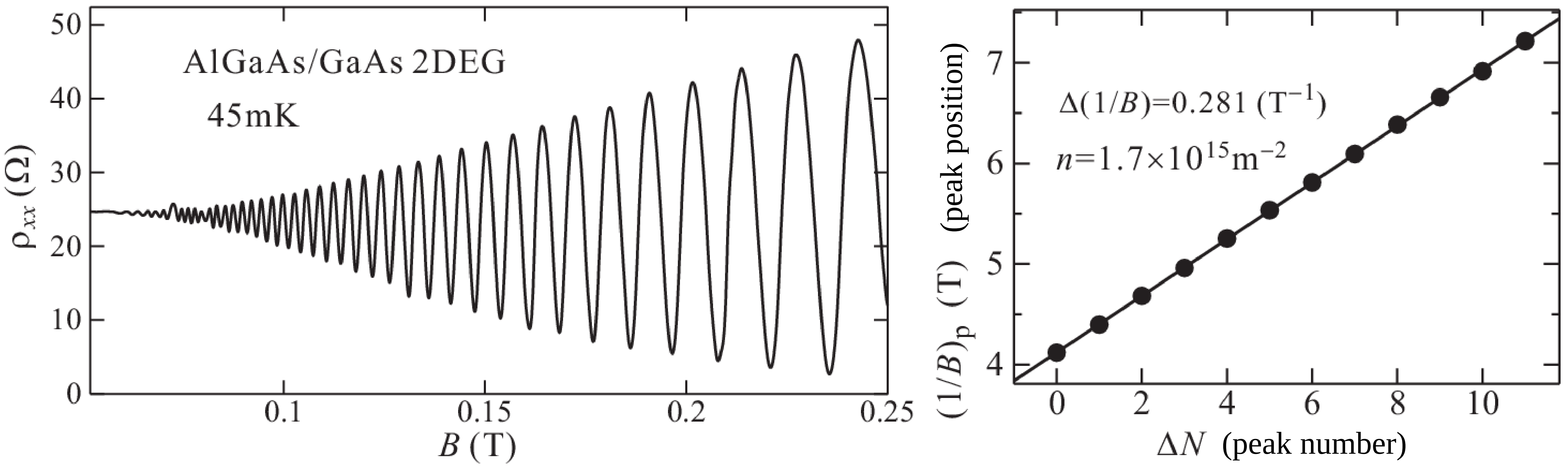
$$\frac{W_x/l^2}{2\pi/W_y} = \frac{S}{2\pi l^2} \quad \rho_L = \frac{1}{2\pi l^2} = \frac{eB}{h} = \frac{B}{\phi_0}$$

$\nu = \dfrac{\phi_0 n_s}{B}$: Filling factor (number of Landau levels filled with electrons)



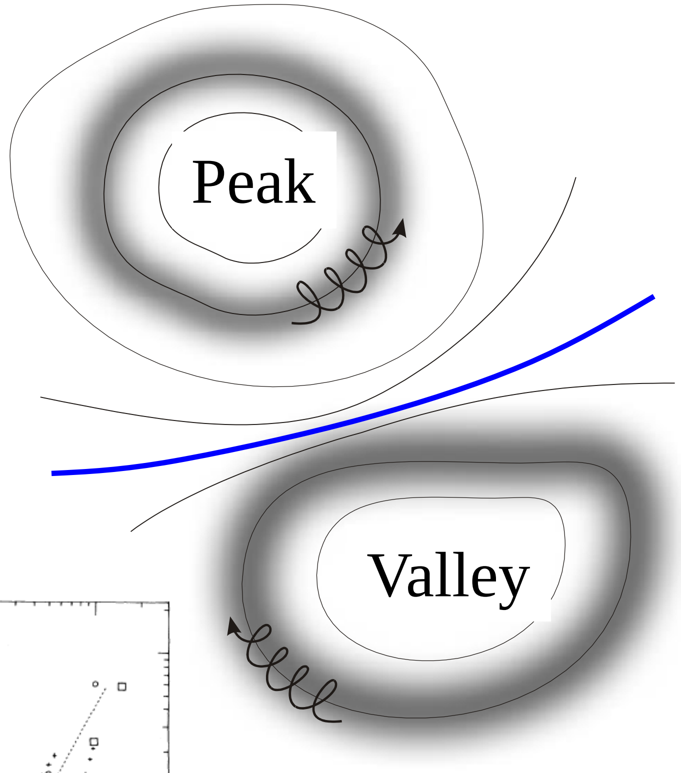


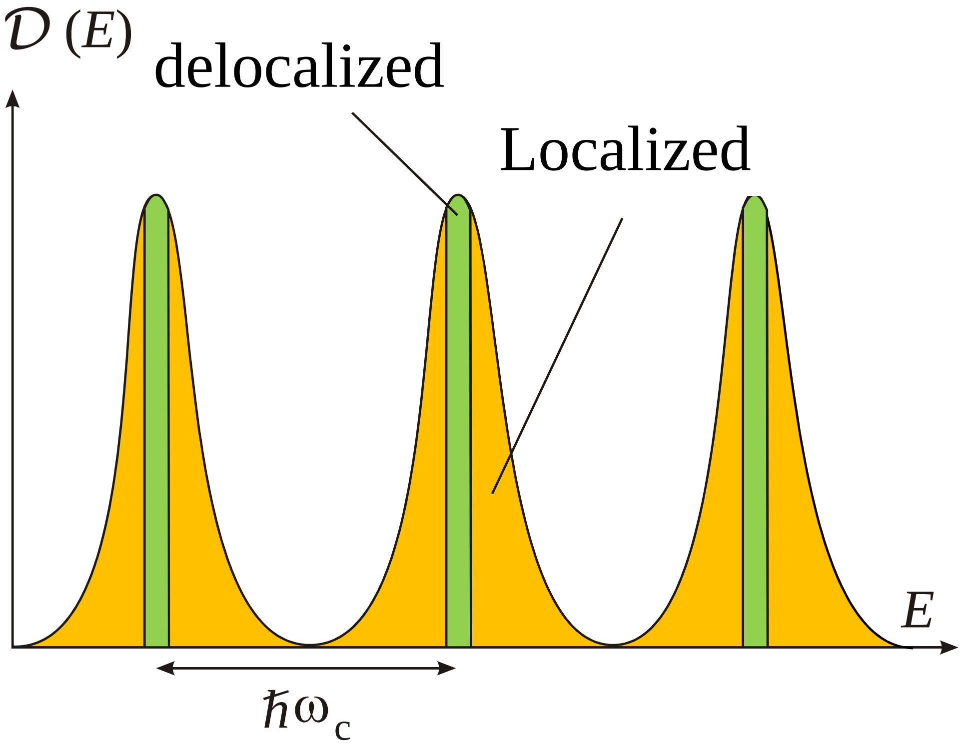
SdH oscillation (example)

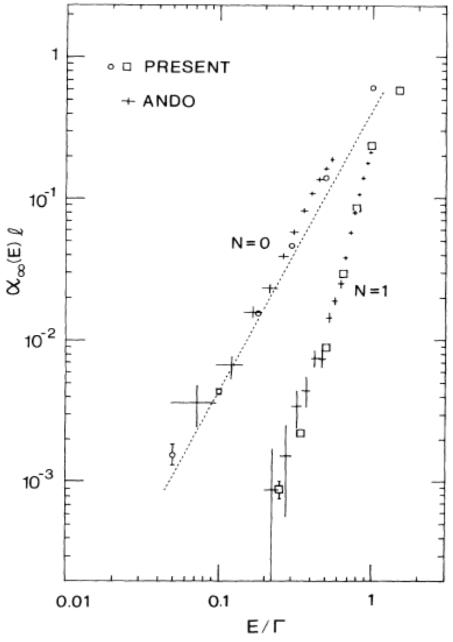


$$n = \frac{2}{\phi_0 \Delta(1/B)} = \frac{4.83 \times 10^{14}}{\Delta(1/B)} \quad (\mathrm{m}^{-2})$$

Localization/delocalization of wavefunctions





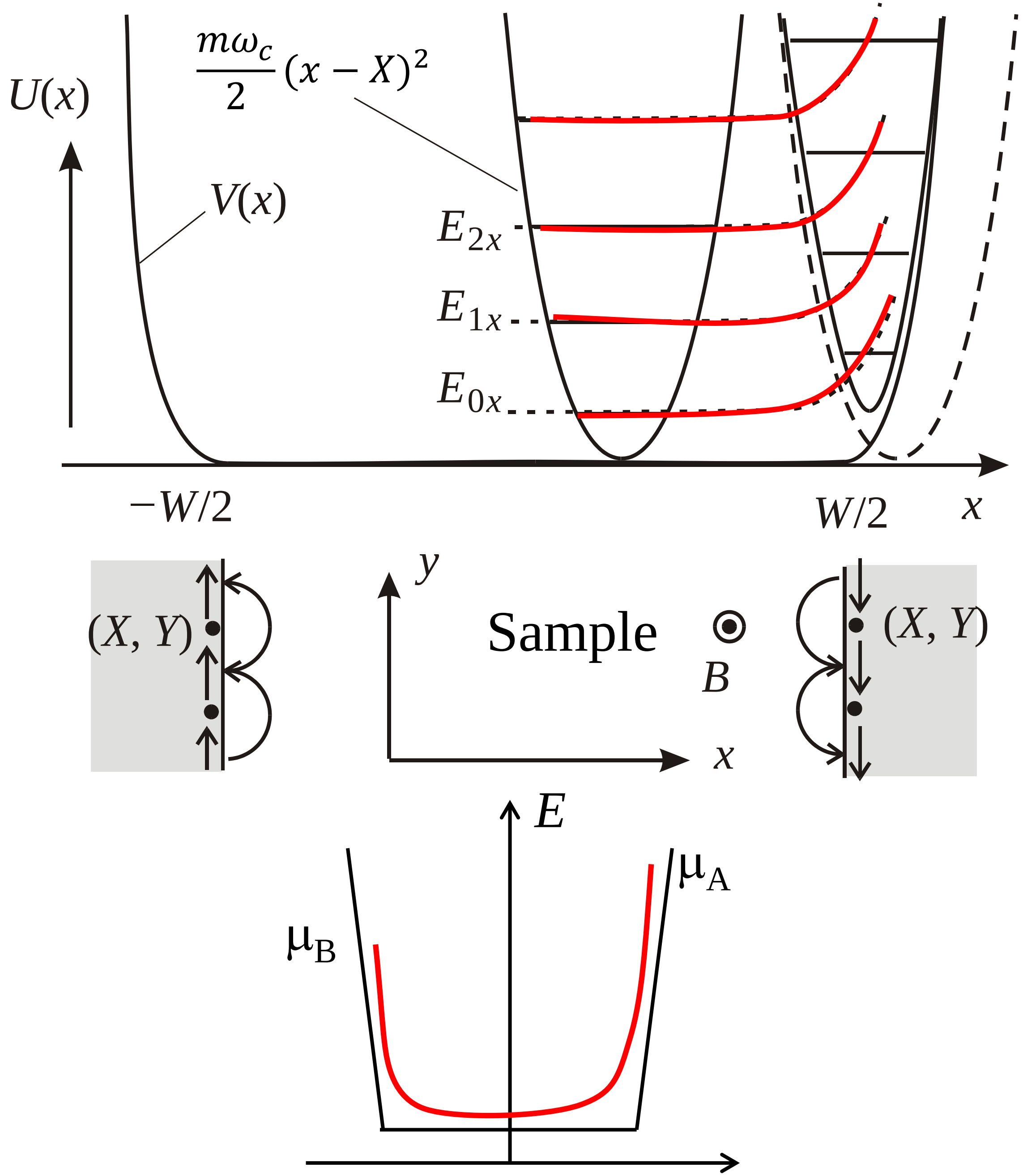


$$\xi(E)^{-1} = \alpha(E) \propto |E - E_N|^s$$

Numerical simulation $s = 2$ for $N = 0$

Aoki & Ando, PRL **54**, 831 (1985).

Edge mode explanation of IQHE



In an edge mode, the group velocity appears because the energy levels varies with *x*.

$$\langle v_y \rangle = \frac{dE}{\hbar dk} = -\frac{l_B^2}{\hbar}\frac{dE}{dX}$$

Current brought by a Landau edge mode

$$J = \int_{X_0}^{X_\mu} \frac{L_y dX}{2\pi l_B^2}\frac{e}{L_y}\langle v_y \rangle = \frac{e}{h}\int \mathrm{d}X \frac{\mathrm{d}E}{\mathrm{d}X} = \frac{e}{h}(\mu - E_0)$$

One dimensional system:

Landauer formula is applicable

$$\sigma_{xy} = \frac{J_y}{V_x} = \frac{e(J_\mathrm{A} - J_\mathrm{B})}{\mu_\mathrm{A} - \mu_\mathrm{B}} = \frac{e^2}{h}$$

Chiral edge mode: No backscattering!

Explanation from topological aspect

Bloch electrons under magnetic field: tight binding model

Translational operator: $T_{\boldsymbol{R}} f(\boldsymbol{r}) = f(\boldsymbol{r}+\boldsymbol{R}), \quad T_{\boldsymbol{R}} = \exp\left(\frac{i}{\hbar}\boldsymbol{R}\cdot\boldsymbol{p}\right)$

Hamiltonian: $\mathscr{H}_0 = -\frac{\hbar^2\nabla^2}{2m} + V(\boldsymbol{r})$

→ simultaneous diagonalization → Bloch states

$$\mathscr{H} = \frac{1}{2m}(\boldsymbol{p}+e\boldsymbol{A})^2 + V(\boldsymbol{r})$$

$\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}(\boldsymbol{r}+\boldsymbol{R}) + \boldsymbol{\nabla} g(\boldsymbol{r})$ does not have translational symmetry

Magnetic translation operator $\quad \boldsymbol{p} \to \boldsymbol{p} + e\boldsymbol{A}$

Symmetric gauge $\quad \boldsymbol{A} = \boldsymbol{B}\times\boldsymbol{r}/2$

$$T_{B\boldsymbol{R}} \equiv \exp\left\{\frac{i}{\hbar}\boldsymbol{R}\cdot\left[\boldsymbol{p}+\frac{e}{2}(\boldsymbol{r}\times\boldsymbol{B})\right]\right\} = T_{\boldsymbol{R}}\exp\left[\frac{ie}{\hbar}(\boldsymbol{B}\times\boldsymbol{R})\cdot\frac{\boldsymbol{r}}{2}\right]$$

$$[\mathscr{H}, T_{B\boldsymbol{R}}] = 0$$

Magnetic Brillouin zone

However $T_{B\boldsymbol{R}a}T_{B\boldsymbol{R}b} = \exp(2\pi i\phi)T_{B\boldsymbol{R}b}T_{B\boldsymbol{R}a}, \quad \phi = \dfrac{eB}{h}ab$

$\phi = p/q \;\; : \;\;$ rational number

Magnetic unit cell: unit vectors $(\boldsymbol{a}, \boldsymbol{b}) \rightarrow$ magnetic unit vectors $(q\boldsymbol{a}, \boldsymbol{b})$

Lattice vector : $R' = n(q\boldsymbol{a}) + m\boldsymbol{b}$ $\quad T_{B\boldsymbol{R}'}$: elements commute

ψ: simultaneously diagonalizes $\mathcal{H}$ and $T_{B\boldsymbol{R}'}$

Magnetic Brillouin zone: $0 \leq k_1 < 2\pi/qa, \; 0 \leq k_2 < 2\pi/b$

$$T_{q\boldsymbol{a}+\boldsymbol{b}}\psi = \exp[i(k_x qa + k_y b)]\psi$$

Magnetic Bloch function: $\psi_{n\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\boldsymbol{r}}u_{n\boldsymbol{k}}(\boldsymbol{r})$

Magnetic Bloch function

$$u_{n\boldsymbol{k}}(x+qa,y)=\exp\left(i\frac{\pi py}{b}\right)u_{n\boldsymbol{k}}(x,y),$$

$$u_{n\boldsymbol{k}}(x,y+b)=\exp\left(-i\frac{\pi px}{qa}\right)u_{n\boldsymbol{k}}(x,y).$$

$$u_{n\boldsymbol{k}}(\boldsymbol{r})=\left|u_{n\boldsymbol{k}(\boldsymbol{r})}\right|\exp[i\theta_{\boldsymbol{k}}(\boldsymbol{r})] \qquad p=-\frac{1}{2\pi}\oint d\boldsymbol{l}\cdot\frac{\partial\theta_{\boldsymbol{k}}(\boldsymbol{r})}{\partial\boldsymbol{l}}$$

Remember k·p approximation $\boldsymbol{p}e^{i\boldsymbol{k}\boldsymbol{r}}=e^{i\boldsymbol{k}\boldsymbol{r}}(\hbar\boldsymbol{k}+\boldsymbol{p})$

$$(\boldsymbol{p}+e\boldsymbol{A})^2e^{i\boldsymbol{k}\boldsymbol{r}}u_{n\boldsymbol{k}}(\boldsymbol{r})=e^{i\boldsymbol{k}\boldsymbol{r}}(\hbar\boldsymbol{k}+\boldsymbol{p}+e\boldsymbol{A})^2u_{n\boldsymbol{k}}(\boldsymbol{r})$$

Schrodinger-like equation for $u_{n\boldsymbol{k}}(\boldsymbol{r})$

$$\mathscr{H}_{\boldsymbol{k}}u_{n\boldsymbol{k}}(\boldsymbol{r})=E_{n\boldsymbol{k}}u_{n\boldsymbol{k}}(\boldsymbol{r}), \qquad \mathscr{H}_{\boldsymbol{k}}=\frac{1}{2m}(-i\hbar\boldsymbol{\nabla}+\hbar\boldsymbol{k}+e\boldsymbol{A})^2+V(\boldsymbol{r})$$

k- dependent Hamiltonian

Kubo formula for σ_{xy}

Ryogo Kubo
1920 - 1995

Electric field along *y*-axis: *E*

$$|\alpha'\rangle = |\alpha\rangle + \sum_{\beta\neq\alpha} \frac{\langle\beta|eEy|\alpha\rangle}{E_\alpha - E_\beta}|\beta\rangle$$

Unperturbed state

$$j_x = \frac{1}{L^2}\sum_\alpha f(E_{\alpha'})\langle\alpha'|\hat{j}_x|\alpha'\rangle = \frac{1}{L^2}\sum_\alpha f(E_\alpha)\sum_{\beta\neq\alpha}\frac{\langle\alpha|(-ev_x)|\beta\rangle\langle\beta|eEy|\alpha\rangle}{E_\alpha - E_\beta} + \text{c.c.}$$

$$\langle\beta|v_y|\alpha\rangle = \langle\beta|\dot{y}|\alpha\rangle = -\frac{i}{\hbar}\langle\beta|[y,\mathscr{H}]|\alpha\rangle = -\frac{i}{\hbar}(E_\alpha - E_\beta)\langle\beta|y|\alpha\rangle$$

$$\sigma_{xy} = \frac{j_x}{E} = \frac{e^2\hbar}{iL^2}\sum_\alpha f(E_\alpha)\sum_\beta \frac{\langle\alpha|v_x|\beta\rangle\langle\beta|v_y|\alpha\rangle}{(E_\alpha - E_\beta)^2} + \text{c.c.}$$

Magnetic Bloch function (II)

Velocity operator: $\boldsymbol{v} = (-i\hbar\boldsymbol{\nabla} + e\boldsymbol{A})/m$

$$u_{n\boldsymbol{k}}(\boldsymbol{r}) \to |n, \boldsymbol{k}\rangle$$

$$\langle n, \boldsymbol{k}|\boldsymbol{v}|m, \boldsymbol{k}'\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'} \int_0^{qa} dx \int_0^b dy u^*_{n\boldsymbol{k}} \boldsymbol{v} u_{m\boldsymbol{k}'} \equiv \delta_{\boldsymbol{k}\boldsymbol{k}'} \langle n|\boldsymbol{v}|m\rangle$$

Normalization: $\int_0^{qa} dx \int_0^b dy |u_{n\boldsymbol{k}}(\boldsymbol{r})|^2 = 1$

$$\langle n|v_x|m\rangle = \frac{1}{\hbar}\left\langle n\left|\frac{\partial \mathscr{H}_{\boldsymbol{k}}}{\partial k_x}\right|m\right\rangle, \quad \langle n|v_y|m\rangle = \frac{1}{\hbar}\left\langle n\left|\frac{\partial \mathscr{H}_{\boldsymbol{k}}}{\partial k_y}\right|m\right\rangle.$$

$$\left\langle n\left|\frac{\partial \mathscr{H}_{\boldsymbol{k}}}{\partial k_j}\right|m\right\rangle = (E_m - E_n)\left\langle n\left|\frac{\partial u_m}{\partial k_j}\right.\right\rangle = -(E_m - E_n)\left\langle\left.\frac{\partial u_n}{\partial k_j}\right|m\right\rangle,$$

$j = x.y$

Kubo conductivity calculated with magnetic Bloch functions

$$\sigma_{xy} = -i\frac{e^2}{\hbar}\sum_{\boldsymbol{k}}\sum_{n} f(E_{n\boldsymbol{k}}) \sum_{m(\neq n)} \left[\frac{\langle n\boldsymbol{k}|\partial\mathscr{H}_{\boldsymbol{k}}/\partial k_x|m\boldsymbol{k}\rangle\langle m\boldsymbol{k}|\partial\mathscr{H}_{\boldsymbol{k}}/\partial k_y|n\boldsymbol{k}\rangle}{(E_{n\boldsymbol{k}}-E_{m\boldsymbol{k}})^2} - \text{c.c.}\right]$$

$$= -i\frac{e^2}{\hbar}\sum_{\boldsymbol{k}}\sum_{n} f(E_{n\boldsymbol{k}}) \sum_{m(\neq n)} \left[\left\langle \frac{\partial u_n}{\partial k_x}\middle| m\right\rangle\left\langle m\middle|\frac{\partial u_n}{\partial k_y}\right\rangle - \left\langle \frac{\partial u_n}{\partial k_y}\middle| m\right\rangle\left\langle m\middle|\frac{\partial u_n}{\partial k_x}\right\rangle\right]$$

$$= \frac{e^2}{h}\frac{2\pi}{i}\sum_{\boldsymbol{k}}\sum_{n} f(E_{n\boldsymbol{k}}) \left[\left\langle \frac{\partial u_n}{\partial k_x}\middle|\frac{\partial u_n}{\partial k_y}\right\rangle - \left\langle \frac{\partial u_n}{\partial k_y}\middle|\frac{\partial u_n}{\partial k_x}\right\rangle\right].$$

Vector field: $\boldsymbol{A}_{n\boldsymbol{k}} = \int d^2\boldsymbol{r}\, u^*_{n\boldsymbol{k}} \boldsymbol{\nabla}_{\boldsymbol{k}} u_{n\boldsymbol{k}} = \langle u_{n\boldsymbol{k}}|\boldsymbol{\nabla}_{\boldsymbol{k}}|u_{n\boldsymbol{k}}\rangle$ Berry connection

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi i}\sum_{E_n<E_{\rm F}}\int_{\rm MBZ} d^2k[\boldsymbol{\nabla}_{\boldsymbol{k}}\times\boldsymbol{A}_{n\boldsymbol{k}}]_{k_z} = \frac{e^2}{h}\frac{1}{2\pi i}\sum_{E_n<E_{\rm F}}\int_{\rm MBZ} d^2k[\text{rot}_{\boldsymbol{k}}\boldsymbol{A}_{n\boldsymbol{k}}]_{k_z}$$

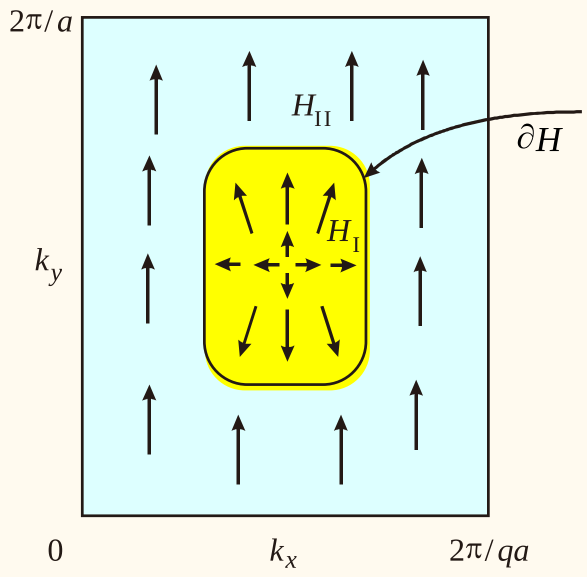
Berry curvature

TKNN Formula

Existence of zero or anomaly

Magnetic Brillouin zone

$$I = \frac{1}{2\pi i}\left[\int_{\mathrm{I}} d^2k[\mathrm{rot}\boldsymbol{A}]_{k_z} + \int_{\mathrm{II}} d^2k[\mathrm{rot}\boldsymbol{A}]_{k_z}\right] = \oint_{\partial H}(\boldsymbol{A}^{\mathrm{II}} - \boldsymbol{A}^{\mathrm{I}})\cdot\frac{d\boldsymbol{k}}{2\pi i}$$



On the boundary ∂H $\quad u^{\mathrm{I}}_{\boldsymbol{k}} = u^{\mathrm{II}}_{\boldsymbol{k}} e^{i\theta(\boldsymbol{k})}$

$$I = \oint_{\partial H}\left[\langle u^{\mathrm{II}}_{\boldsymbol{k}}|\boldsymbol{\nabla}_{\boldsymbol{k}}|u^{\mathrm{II}}_{\boldsymbol{k}}\rangle + (i\boldsymbol{\nabla}_{\boldsymbol{k}}\theta)\langle u^{\mathrm{II}}_{\boldsymbol{k}}|u^{\mathrm{II}}_{\boldsymbol{k}}\rangle - \langle u^{\mathrm{II}}_{\boldsymbol{k}}|\boldsymbol{\nabla}_{\boldsymbol{k}}|u^{\mathrm{II}}_{\boldsymbol{k}}\rangle\right]\cdot\frac{d\boldsymbol{k}}{2\pi i}$$

$$= \frac{\Delta_{\partial H}\theta}{2\pi} = \nu_{\mathrm{C}}$$

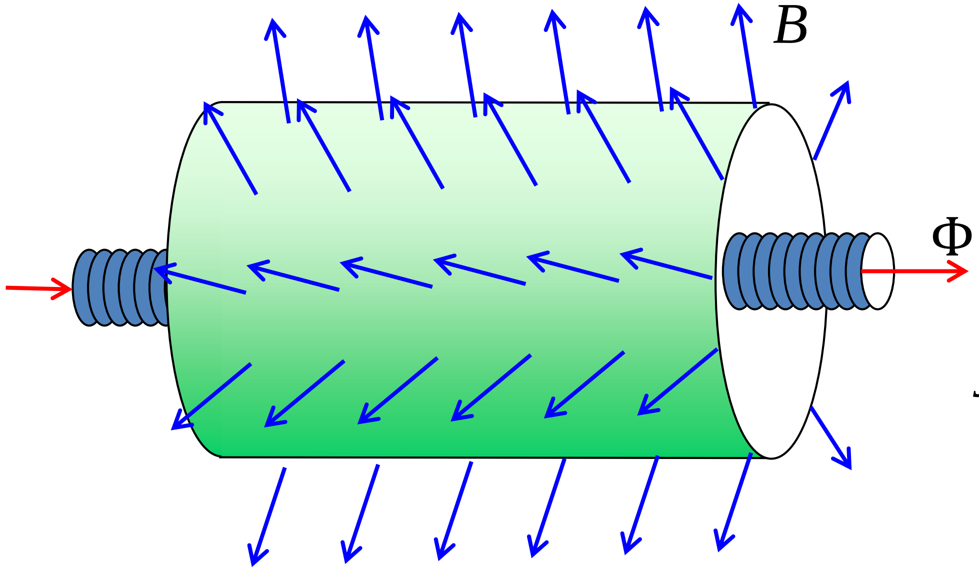
: Chern number (integer)

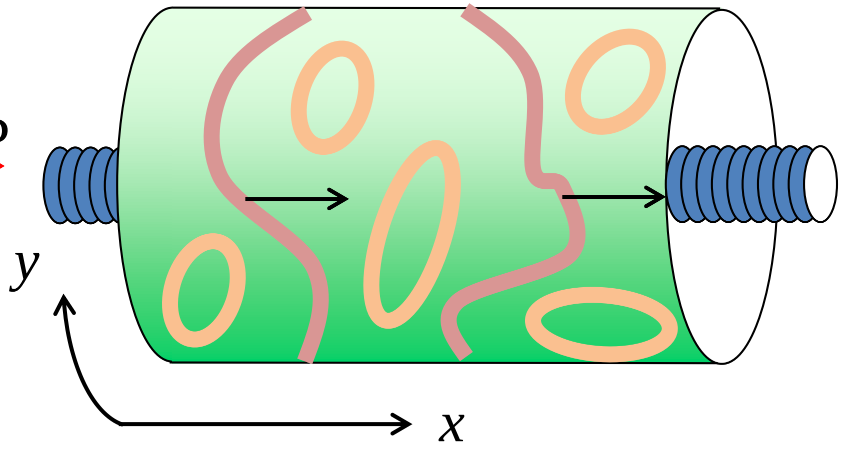
Topological invariant

$$\sigma_{xy} = \nu_{\mathrm{C}}\frac{e^2}{h}$$

Thouless-Kohmoto-Nightingale-den Nijs (TKNN) Formula

Laughlin's discussion





R. Laughlin, Phys. Rev. B **23**, 5632 (1981).

Robert Laughlin

Landau gauge $\boldsymbol{A} = (0, Bx - \Phi/L_y) = (0, B(x - \Phi/L_y B))$

Magnetic flux Φ : *X* shift $X \to X + \dfrac{\Phi}{L_y B} \ : \ \dfrac{\Phi}{\phi_0}\dfrac{L_x}{N_\mathrm{L}} \quad (N_\mathrm{L} \equiv n_\mathrm{L} L_x L_y)$

$$j_y = \frac{J_y}{L_x} = \frac{1}{L_x}\frac{\partial E_{L_x}}{\partial \Phi} \quad \left(cf.\ E = \frac{L}{2}J^2,\ \Phi = LJ\right)$$

$$= \frac{1}{L_x}\frac{\Delta E_{L_x}}{\Delta \Phi} = \frac{1}{L_x}\left(-e\mathcal{E}_x \frac{L_x}{N_\mathrm{L}}\right)\frac{N_e}{\phi_0} = \nu\frac{e^2}{h}\mathcal{E}_x$$

Chern number =1

Summary of "topological aspect"

(a) In 2D system under magnetic field: magnetic Bloch functions, magnetic Brillouin zone

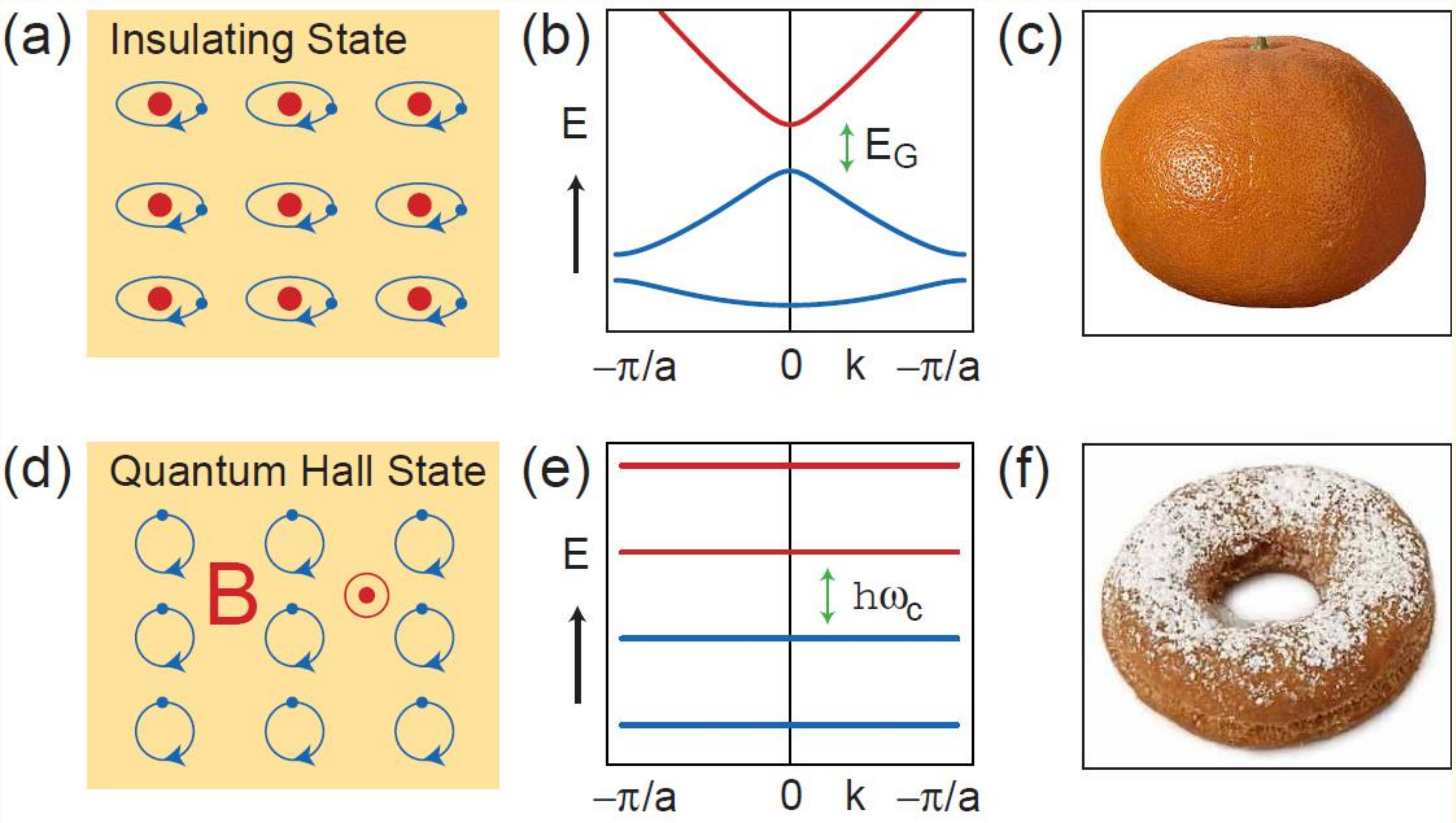
(b) Kubo formula for Hall conductivity: matrix elements of velocity operator

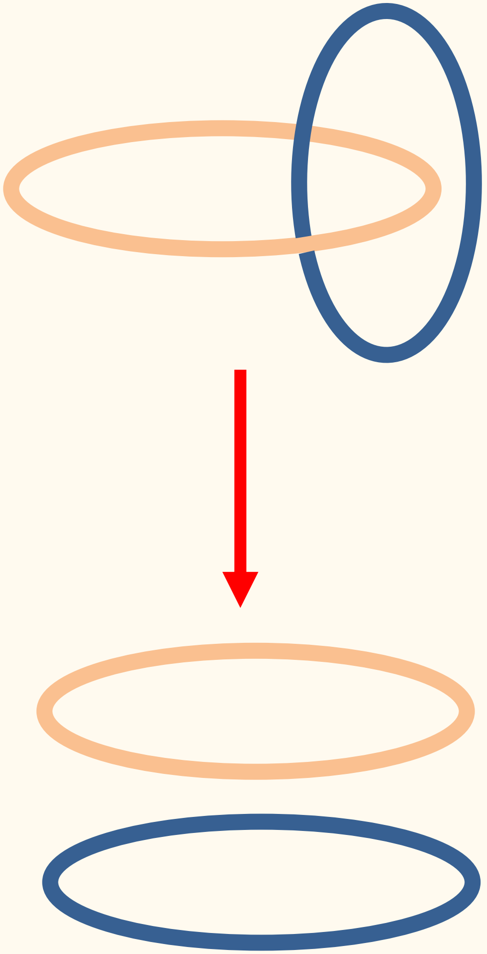
(c) From (a) and (b) Hall conductivity is obtained as the integration of Berry curvature over magnetic Brillouin zone

(d) TKNN formula: Chern number (topological invariant) times quantum conductance

(e) Chern number is integer (due to single-valuedness of atomic part) and non-zero in quantum Hall system (Laughlin's discussion)

Bulk-Edge correspondence

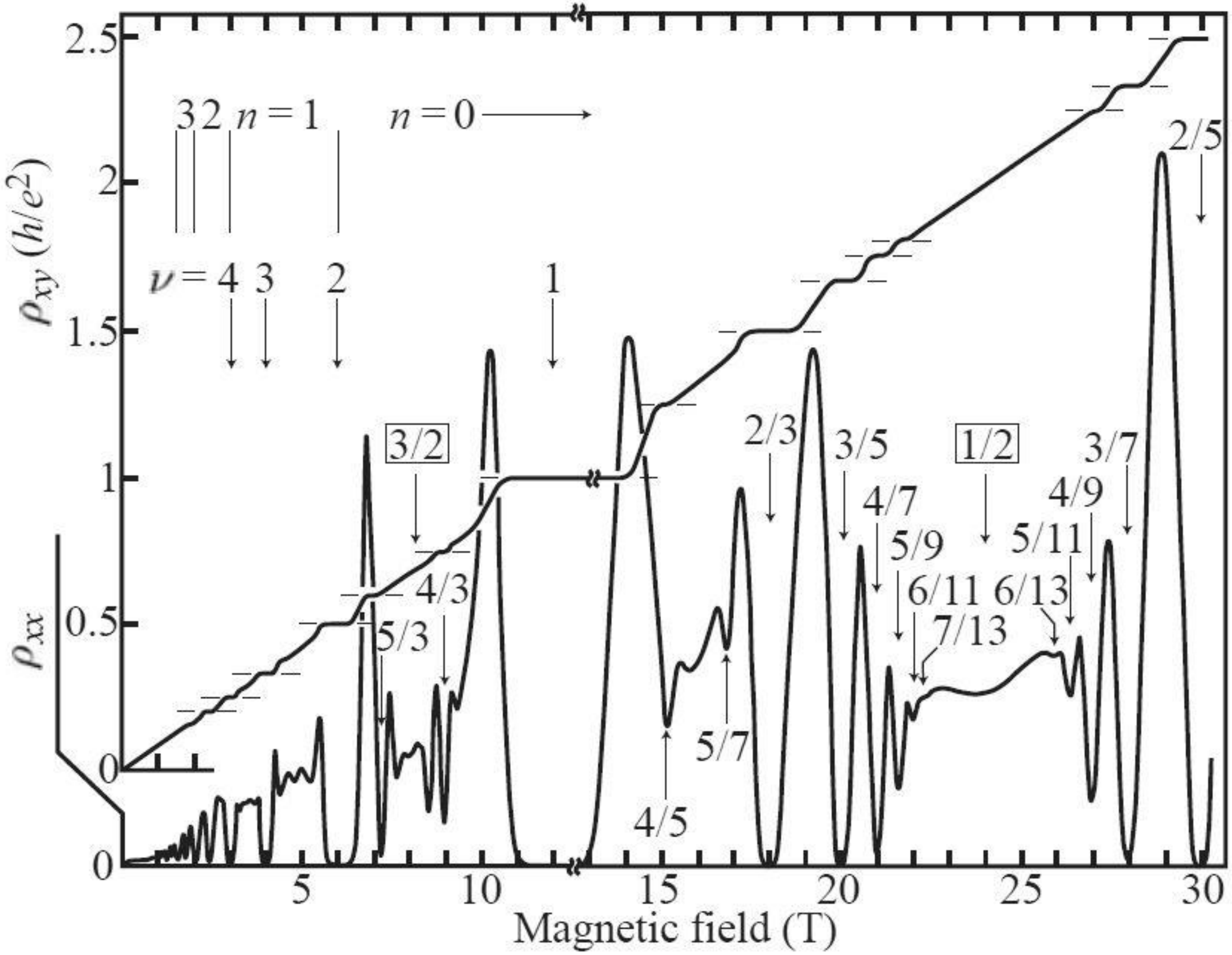




Hasan & Kane, Rev. Mod. Phys. **82**, 3045 (2010).

Transition between bands with different Chern number only can attained through energy gap collapse.

Fractional quantum Hall effect



Laughlin state

$$\psi_q(z_1, \cdots, z_{N_e}) = \prod_{i>j} (z_i - z_j)^q \exp\left(-\sum_i \frac{|z_i|^2}{4}\right)$$